Rainbow Numbers of $\mathbb{Z}_n$ for $a_1x_1 + a_2x_2 + a_3x_3 = b$

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Abstract

An exact r-coloring of a set S is a surjective function $c : S \to [r]$. The rainbow number of a set S for equation eq is the smallest integer r such that every exact r-coloring of S contains a rainbow solution to eq. In this paper, the rainbow number of $\mathbb{Z}_p$, for p prime and the equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ is determined. The rainbow number of $\mathbb{Z}_n$, for a natural number n, is determined under certain conditions.

1 Introduction

Let c be a coloring of set S. A subset $X \subseteq S$ is rainbow if each element of X is colored distinctly. For example, color $[n] = \{1, 2, \ldots, n\}$ and consider solutions to the equation $x_1 + x_2 = x_3$. If each element of a solution $\{a, b, a + b\} \subseteq [n]$ is colored distinctly, that solution is rainbow. One of the first papers to investigate rainbow arithmetic progressions is [7], where Jungić et al. showed that colorings with each color used equally yield rainbow arithmetic progressions. In [7], only 3-term arithmetic progressions are considered which are also solutions to $x_1 + x_2 = 2x_3$. In [1], Axenovich and Fon-Der-Flaass showed that no 5-colorings avoid rainbow 3-term arithmetic progressions. A few articles investigated the anti-van der Waerden number, which is the fewest number of colors need to guarantee a rainbow arithmetic progression. For example, Butler et al. established, in [4], bounds for the anti-van der Waerden number when coloring $[n]$ and some exact values when coloring $\mathbb{Z}_n$. Later, Berikkyzy, Schulte, and Young determined, in [2], the anti-van der Waerden number for $[n]$ in the case of 3-term arithmetic progressions.

Some of this work was generalized to graphs and abelian groups. Montejano and Serra investigated, in [9], rainbow-free colorings of abelian groups when considering arithmetic progressions. Similarly, rainbow arithmetic progressions in finite abelian groups were studied by co-author Young, in [11], where the anti-van der Waerden numbers were connected to the order of the group. When arithmetic progressions were extended to graphs, Rehm, Schulte, and Warnberg showed, in [10], the anti-van der Waerden numbers on graph products is either 3 or 4.

Generalizing the equation $x_1 + x_2 = 2x_3$, Bevilacqua et al., in [3], considered $x_1 + x_2 = kx_3$ on $\mathbb{Z}_n$. The rainbow number of $\mathbb{Z}_n$ was determined for these equations when $k = 1$ or $k = p$ where p is prime. These results served as motivation for this paper where the equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ will be considered over $\mathbb{Z}_p$ and $\mathbb{Z}_n$ with p prime. From now on $a_1x_1 + a_2x_2 + a_3x_3 = b$ will be denoted by $\text{eq}(a_1, a_2, a_3, b)$. This paper establishes the rainbow number, also known as the anti-van der Waerden number, of $\mathbb{Z}_n$ for $\text{eq}(a_1, a_2, a_3, b)$ for some equations. One important result that will be used is Huicochea and Montejano's characterization, in [6], of all rainbow-free exact 3-colorings of $\mathbb{Z}_p$ for $\text{eq}(a_1, a_2, a_3, b)$ for all primes p.

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1.1 Preliminaries

An r-*coloring* of a set S is a function $c : S \to [r]$ and an r-coloring is *exact* if c is surjective. Note that an exact r-coloring yields a partition of S into r disjoint color classes. This paper will focus on the linear equation $\text{eq}(a_1, a_2, a_3, b)$ given by

$$a_1x_1 + a_2x_2 + a_3x_3 = b \tag{1}$$

and r-colorings of $\mathbb{Z}_n$. An ordered set (s_1, s_2, s_3) is called a *solution* to $\text{eq}(a_1, a_2, a_3, b)$ in $\mathbb{Z}_n$ if $a_1s_1 + a_2s_2 + a_3s_3 \equiv b \bmod n$. Throughout the paper $=$ will be used instead of $\equiv$, and the mod n will not be used unless the context requires clarification.

If c is an r-coloring of $\mathbb{Z}_n$, then a *rainbow solution* in $\mathbb{Z}_n$ to $\text{eq}(a_1, a_2, a_3, b)$ is a solution such that $|\{c(s_1), c(s_2), c(s_3)\}| = 3$, i.e. each member of the solution has been assigned a distinct color by c. A coloring c of $\mathbb{Z}_n$ is *rainbow-free* for $\text{eq}(a_1, a_2, a_3, b)$ if there are no rainbow solutions.

The rainbow number of $\mathbb{Z}_n$ for equation $eq = \text{eq}(a_1, a_2, a_3, b)$, denoted $\text{rb}(\mathbb{Z}_n, eq)$, is the smallest positive integer r such that every exact r-coloring of $\mathbb{Z}_n$ has a rainbow solution for eq. If there are no rainbow solutions to eq in an exact n-coloring of $\mathbb{Z}_n$, then the convention will be that $\text{rb}(\mathbb{Z}_n, eq) = n + 1$. Since rainbow solutions to eq require three colors, then $\text{rb}(\mathbb{Z}_n, eq) \geq 3$, for all $n \geq 2$.

The following tools will be used throughout the paper. Given a set $S \subseteq \mathbb{Z}_n$ and $d, t \in \mathbb{Z}_n$, the sets $S + t = \{s + t \mid s \in S\}$ and $dS = \{ds \mid s \in S\}$ are called the t-*translation* and d-*dilation* of S, respectively. If the multiplicative inverse of $a \in \mathbb{Z}_n$ exists, denote the inverse by a^{-1}. The set of all these invertible elements forms a group under multiplication, and it is denoted by $\mathbb{Z}_n^*$. For $d \in \mathbb{Z}_n^*$, let $\langle d \rangle$ be the multiplicative subgroup of $\mathbb{Z}_n^*$ generated by d and $\langle d_1, \ldots, d_k \rangle$ be multiplicatively generated by the d_i's. A subset $S \subseteq \mathbb{Z}_n$ is $\langle d \rangle$-*periodic* if $S = dS$ and a set is called *symmetric* if it is $\langle -1 \rangle$-periodic. For ease of reading, the related results from [6] are referenced below.

Theorem 1.1. *([6, Theorem 3]). Let A, B and C be the color classes of an exact 3-coloring of $\mathbb{Z}_p$ such that $1 \leq |A| \leq |B| \leq |C|$. The coloring is rainbow-free for $\text{eq}(1, 1, -c, 0)$ if and only if, under dilation, one of the following holds true:*

1) $A = \{0\}$, with both B and C symmetric $\langle c \rangle$-periodic subsets.

2) $A = \{1\}$ for

a) $c = 2$, with $(B - 1)$ and $(C - 1)$ symmetric $\langle 2 \rangle$-periodic subsets;

b) $c = -1$, with $(B \backslash \{-2\}) + 2^{-1}$ and $(C \backslash \{-2\}) + 2^{-1}$ symmetric subsets.

3) $|A| \geq 2$, for $c = -1$, with A, B and C arithmetic progressions with difference 1, such that $A = \{i\}_{i=t_1}^{t_2-1}$, $B = \{i\}_{i=t_2}^{t_3-1}$, and $C = \{i\}_{i=t_3}^{t_1-1}$, where $(t_1 + t_2 + t_3) = 1$ or 2.

Theorem 1.2. *([6, Theorem 6]). Let A, B and C be the color classes of an exact 3-coloring of $\mathbb{Z}_p$ such that $1 \leq |A| \leq |B| \leq |C|$. The coloring is rainbow-free for $\text{eq}(a_1, a_2, a_3, b)$, with some $a_i \neq a_j$, if and only if $A = \{s\}$ with $s(a_1 + a_2 + a_3) = b$, and both B and C are sets invariant under six specific transformations.*

Corollary 1.3. *([6, Corollary 8]). Every exact 3-coloring of $\mathbb{Z}_p$ contains a rainbow solution of $\text{eq}(a_1, a_2, a_3, b)$, with some $a_i \neq a_j$, if and only if one of the following holds true:*

1) $a_1 + a_2 + a_3 = 0 \neq b$,

2) $|\langle d_1, \ldots, d_6 \rangle| = p - 1$,

where $d_1 = -a_3a_1^{-1}$, $d_2 = -a_2a_1^{-1}$, $d_3 = -a_1a_2^{-1}$, $d_4 = -a_3a_2^{-1}$, $d_5 = -a_1a_3^{-1}$, and $d_6 = -a_2a_3^{-1}$.

Note that Theorem 1.4 is the same as the case when $b = 0$ and $c = -1$ in Theorem 1.1. It is included for completion.

Theorem 1.4. *[6, Theorem 5]. Let A, B and C be the color classes of an exact 3-coloring of $\mathbb{Z}_p$ with $p \geq 3$ and $1 \leq |A| \leq |B| \leq |C|$. The coloring is rainbow-free for $\text{eq}(1, 1, 1, b)$ if and only if one of the following holds true:*

1) $A = \{s\}$ *with both* $(B\backslash\{b-2s\}) + (s-b)2^{-1}$ *and* $(C\backslash\{b-2s\}) + (s-b)2^{-1}$ *symmetric sets.*

2) $|A| \geq 2$, *and all* A, B *and* C *are arithmetic progressions with the same common difference* d, *so that* $d^{-1}A = \{i\}_{i=t_1}^{t_2-1}$, $d^{-1}B = \{i\}_{i=t_2}^{t_3-1}$, *and* $d^{-1}C = \{i\}_{i=t_3}^{t_1-1}$ *satisfy* $t_1 + t_2 + t_3 \in \{1 + d^{-1}b, 2 + d^{-1}b\}$.

Lemma 1.5 will be used to extend results for rainbow numbers of equations where $b = 0$ to equations where $b \neq 0$.

Lemma 1.5. *For* $a_1, a_2, a_3 \in \mathbb{Z}_n$ *let* $a = a_1 + a_2 + a_3$ *and suppose that* $a \in \mathbb{Z}_n^*$. *There exists a rainbow-free* k*-coloring of* $\mathbb{Z}_n$ *for* $a_1x_1 + a_2x_2 + a_3x_3 = b$ *if and only if there exists a rainbow-free* k*-coloring of* $\mathbb{Z}_n$ *for* $a_1x_1 + a_2x_2 + a_3x_3 = 0$.

Proof. Define $T : \mathbb{Z}_n \to \mathbb{Z}_n$ by $T(x) = x - ba^{-1}$. Suppose (s_1, s_2, s_3) is a solution to $a_1x_1 + a_2x_2 + a_3x_3 = b$. Applying the one-to-one transformation T to (s_1, s_2, s_3) gives:

$$\begin{aligned} a_1T(s_1) + a_2T(s_2) + a_3T(s_3) &= a_1(s_1 - ba^{-1}) + a_2(s_2 - ba^{-1}) + a_3(s_3 - ba^{-1}) \\ &= a_1s_1 + a_2s_2 + a_3s_3 + (a_1 + a_2 + a_3)(-ba^{-1}) \\ &= b + a(-ba^{-1}) \\ &= 0. \end{aligned}$$

Similarly, if $(T(s_1), T(s_2), T(s_3))$ is a solution to $a_1x_1 + a_2x_2 + a_3x_3 = 0$, then (s_1, s_2, s_3) is a solution to $a_1x_1 + a_2x_2 + a_3x_3 = b$. This gives a one-to-one correspondence between solutions of the two equations. $\square$

This paper is organized as follows. First, $\mathrm{rb}(\mathbb{Z}_p, \mathrm{eq}(a_1, a_2, a_3, b))$, is determined in Section 2. The main result in this section, Theorem 2.3, states that the rainbow number of $\mathbb{Z}_p$ is either 3 or 4. In Section 3, the rainbow number of $\mathbb{Z}_n$ is computed for a natural number n. The main result in this section, Theorem 3.13, shows that for $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_\ell^{\alpha_\ell}$

$$\mathrm{rb}(\mathbb{Z}_n, \mathrm{eq}(a_1, a_2, a_3, b)) = 2 + \sum_{k=1}^{\ell} \left[\alpha_k(\mathrm{rb}(\mathbb{Z}_{p_k}, \mathrm{eq}(a_1, a_2, a_3, b) - 2)\right],$$

under certain conditions. To prove this result, Section 3 includes leading lemmas and theorems such as finding the rainbow number $\mathrm{rb}(\mathbb{Z}_{2^\alpha}, \mathrm{eq}(a_1, a_2, a_3, b))$ in Theorem 3.1, establishing the right hand side of the above equation as a lower bound in Corollary 3.4, and as an upper bound in Corollary 3.12.

2 Rainbow Numbers of $\mathbb{Z}_p$

This section establishes the rainbow number for the equation $\mathrm{eq}(a_1, a_2, a_3, b)$ over $\mathbb{Z}_p$ where p is a prime. Under certain conditions, Lemma 2.1 establishes that if two elements in a solution are the same, then all three are the same. This fact was mentioned in [6] without proof and has been included for completion.

Lemma 2.1. *If* $a_1s_1 + a_2s_2 + a_3s_3 = 0$ *over* $\mathbb{Z}_p$ *with* $|\{s_1, s_2, s_3\}| < 3$, $a_1 + a_2 + a_3 = 0$ *and* $a_1a_2a_3 \in \mathbb{Z}_p^*$, *then* $s_1 = s_2 = s_3$.

Proof. If $s_1 = s_2 = s_3$ the proof is complete. Without loss of generality, assume $s_1 = s_2$. Observe $a_1 + a_2 + a_3 = 0$ implies $a_3 = -a_1 - a_2$, and therefore $a_1s_1 + a_2s_1 + a_3s_3 = (a_1 + a_2)(s_1 - s_3) = 0$.

Since $\mathbb{Z}_p$ has no zero divisors this gives $a_1 + a_2 = 0$ or $s_1 - s_3 = 0$. Note $a_1 + a_2 = 0$ along with $a_1 + a_2 + a_3 = 0$ gives $a_3 = 0$ which contradicts $a_1a_2a_3 \neq 0$. Therefore, $s_1 - s_3 = 0$ and so $s_1 = s_3$ and $|\{s_1, s_2, s_3\}| = 1$. $\square$

If $p = 2$, by convention $\mathrm{rb}(\mathbb{Z}_2, \mathrm{eq}) = 3$. The case when $p = 3$ is handled next.

Proposition 2.2. *For all $a_1, a_2, a_3, b \in \mathbb{Z}_3$,*

$$\mathrm{rb}(\mathbb{Z}_3, \mathrm{eq}(a_1,a_2,a_3,b)) = \begin{cases} 3 & \text{if } b = 0 \text{ and } a_i = a_j, \quad \text{for some } i \neq j \\ & \text{or } b \neq 0 \text{ and } a_i \neq a_j, \quad \text{for some } i \neq j, \\ 4 & \text{otherwise.} \end{cases}$$

Proof. Note there is only one way (up to isomorphism) to color $\mathbb{Z}_3$ with three distinct colors. Suppose eq has a rainbow solution and, without loss of generality, assume a solution is $(1,2,0)$. For $\mathrm{eq}(a_1,a_2,a_3,0)$, $a_1 + 2a_2 = 0$ implies $a_1 = a_2$. It then follows that a rainbow solution will exist if and only if $a_i = a_j$ for some $i \neq j$, giving $\mathrm{rb}(\mathbb{Z}_3, \mathrm{eq}(a_1,a_2,a_3,0) = 3$. If the a_i's are all distinct, by standard convention, $\mathrm{rb}(\mathbb{Z}_3, \mathrm{eq}(a_1,a_2,a_3,0)) = 4$.

Now consider $\mathrm{eq}(a_1,a_2,a_3,b)$ for $b \neq 0$. The solution $(1,2,0)$ gives $a_1 - a_2 = b$. Since $b \neq 0$, then $a_1 \neq a_2$. It then follows that $\mathrm{rb}(\mathbb{Z}_3, \mathrm{eq}(a_1,a_2,a_3,b)) = 3$ if $a_i \neq a_j$ for some $i \neq j$. Otherwise, $\mathrm{rb}(\mathbb{Z}_3, \mathrm{eq}(a_1,a_2,a_3,b)) = 4$. □

Next, the case $p \geq 5$ will be discussed. Theorem 2.3 shows that the rainbow number of $\mathrm{eq}(a_1,a_2,a_3,b)$ is either 3 or 4 depending on the different variations of a_1, a_2, a_3 and b. The following theorem also uses notation established in Corollary 1.3.

Theorem 2.3. *Let $a_1, a_2, a_3, b \in \mathbb{Z}_p$ with some $a_i \neq a_j$ and $a_1a_2a_3 \in \mathbb{Z}_p^*$ for $p \geq 5$, then*

$$\mathrm{rb}(\mathbb{Z}_p, \mathrm{eq}(a_1,a_2,a_3,b)) = \begin{cases} 3 & \text{if } |\langle d_1, d_2, \ldots, d_6 \rangle| = p-1 \\ & \text{or } a_1 + a_2 + a_3 = 0 \neq b, \\ 4 & \text{otherwise.} \end{cases}$$

Proof. The proof follows by case analysis. First, define $eq = \mathrm{eq}(a_1,a_2,a_3,b)$.

Case 1: $|\langle d_1, d_2, \ldots, d_6 \rangle| = p-1$ or $a_1 + a_2 + a_3 = 0 \neq b$

The conditions in this case are the conditions of Corollary 1.3, thus $\mathrm{rb}(\mathbb{Z}_p, eq) \leq 3$ and $\mathrm{rb}(\mathbb{Z}_p, eq) = 3$.

Case 2: $|\langle d_1, d_2, \ldots, d_6 \rangle| < p-1$ and $a_1 + a_2 + a_3 \neq 0$

By Corollary 1.3, there exists a rainbow-free 3-coloring which implies $\mathrm{rb}(\mathbb{Z}_p, eq) \geq 4$. Since $a_1 + a_2 + a_3 \neq 0$ there is a unique $s \in \mathbb{Z}_p$ such that $s(a_1 + a_2 + a_3) = b$. Suppose there is a 4-coloring of $\mathbb{Z}_p$ with color classes A, B, C, and D such that $s \in A$. Create a 3-coloring with color classes $A \cup B$, C, and D. By construction, s is not in a color class by itself. Theorem 1.2 now guarantees there is a rainbow solution in this 3-coloring which corresponds to a rainbow solution in the 4-coloring. Thus, $\mathrm{rb}(\mathbb{Z}_p, eq) \leq 4$ and hence, $\mathrm{rb}(\mathbb{Z}_p, eq) = 4$.

Case 3: $|\langle d_1, d_2, \ldots, d_6 \rangle| < p-1$, $a_1 + a_2 + a_3 = 0$, and $b = 0$

Since $|\langle d_1, d_2, \ldots, d_6 \rangle| < p-1$ and $b = 0$, both conditions in Corollary 1.3 fail; hence, $\mathrm{rb}(\mathbb{Z}_p, eq) \geq 4$. Note that in this case, every $s \in \mathbb{Z}_p$ satisfies $s(a_1 + a_2 + a_3) = b$. To show that $\mathrm{rb}(\mathbb{Z}_p, eq) \leq 4$, consider a 4-coloring of $\mathbb{Z}_p$.

Case 3.1: At most two color classes have size one.

Combine the two smallest color classes to make a 3-coloring that has no color classes of size one. By Theorem 1.2, this 3-coloring contains a rainbow solution. Thus, the original 4-coloring contains a rainbow solution, which implies $\mathrm{rb}(\mathbb{Z}_p, eq) \leq 4$.

Note, if there are at least three color classes of size one, then the argument used in Case 3.1 does not hold. Essentially, combining the two smallest color classes will give a 3-coloring that has a color class with one element.

Case 3.2: At least three color classes have size one.

Let $A = \{s_1\}$ and $B = \{s_2\}$ be two of the three color classes of size one. Let $s_3 = a_3^{-1}(-a_1s_1 - a_2s_2)$, then (s_1, s_2, s_3) is a solution. Since $s_1 \neq s_2$, by Lemma 2.1, then s_1, s_2, s_3 are distinct. Therefore, (s_1, s_2, s_3) is a rainbow solution. Thus, the 4-coloring contains a rainbow, which implies $\mathrm{rb}(\mathbb{Z}_p, eq) \leq 4$. □

Note that Theorem 2.3 considered equations where $a_i \neq a_j$ for some $i \neq j$. For the remainder of this section it is assumed that $a_1 = a_2 = a_3$. To handle equations of this type, Theorem 1.1 and Lemma 2.4 are essential.

Lemma 2.4. *Suppose sets A, B, C, and D partition $\mathbb{Z}_p$. The sets $A \cup B$, $A \cup C$, $A \cup D$, B, C, and D cannot all be arithmetic progressions with common difference $d \neq 0$.*

Proof. For the sake of contradiction, suppose $A \cup B$, $A \cup C$, $A \cup D$, B, C, and D are all arithmetic progressions with common difference d. Define $B = \{\beta, \beta + d, \ldots, \beta + kd\}$. Since B and $A \cup B$ are both arithmetic progressions with the same common difference, then A contains $\beta - d$ or $\beta + (k+1)d$. Similarly, this applies to C and $A \cup C$ and applies to D and $A \cup D$. However, this implies that B, C, and D are not pairwise disjoint, a contradiction. □

It will be shown in Lemma 2.7 that an arithmetic progression D of $\mathbb{Z}_p$, with $2 \leq |D| \leq p - 2$, of common difference d can only be viewed as an arithmetic progression of common difference $\pm d$. Consider the interval notation $[x, y]$, for $x < y$, as defined in [5] as follows. For $x, y \in \mathbb{Z}_p$, let $k = y - x$ and $[x, y] := \{x + i \in \mathbb{Z}_p \mid 0 \leq i \leq k\}$. The author in [5] defines an arithmetic progression of common difference r and length $k + 1$ as the dilated interval $r[x, y]$. The set of such arithmetic progressions is denoted, in [5], by $\mathrm{AP}(r)$.

Lemma 2.5. *[5, Lemma 3.4] Let X be a subset of $\mathbb{Z}_p$ such that $2 \leq |X| \leq p - 2$ and $r, t \in \mathbb{Z}_p^*$. If $X, tX \in \mathrm{AP}(r)$, then $t \in \{\pm 1\}$.*

This lemma can be generalized as follows.

Corollary 2.6. *Let X be a subset of $\mathbb{Z}_p$ such that $2 \leq |X| \leq p - 2$. If $tX, t'X \in \mathrm{AP}(r)$ for $t, t' \in \mathbb{Z}_p^*$, then $t' \in \{\pm t\}$.*

Proof. Let $tX = r[x, y]$ and $t'X = r[x', y']$. Then $X = t^{-1}r[x, y] \in \mathrm{AP}(t^{-1}r)$ and $t^{-1}t'X = t^{-1}r[x', y'] \in \mathrm{AP}(t^{-1}r)$. By Lemma 2.5, $t^{-1}t' \in \{\pm 1\}$, and hence $t' \in \{\pm t\}$. □

Lemma 2.7. *Let D be a subset of $\mathbb{Z}_p$ such that $2 \leq |D| \leq p - 2$ and $d, r \in \mathbb{Z}_p^*$. If D is an arithmetic progression with difference d and D is an arithmetic progression with difference r, then $r \in \{\pm d\}$.*

Proof. Let $D = \{x, x+d, \ldots, x+kd\}$ and $D = \{x', x'+r, \ldots, x'+kr\}$. Then $D = d[d^{-1}x, d^{-1}x+k] \in \mathrm{AP}(d)$ and $D = r[r^{-1}x', r^{-1}x' + k] \in \mathrm{AP}(r)$. Multiplying these two dilated intervals by r and d, respectively, gives $rD = rd[d^{-1}x, d^{-1}x + k] \in \mathrm{AP}(rd)$ and $dD = rd[r^{-1}x', r^{-1}x' + k] \in \mathrm{AP}(rd)$. Thus rD and dD are both in $\mathrm{AP}(rd)$. Applying Corollary 2.6 gives $r \in \{\pm d\}$. □

Theorem 2.8. *If $a \in \mathbb{Z}_p^*$, $b \in \mathbb{Z}_p$, and $p \geq 5$, then* $\mathrm{rb}(\mathbb{Z}_p, \mathrm{eq}(a, a, a, b)) = 4$.

Proof. Since $p \geq 5$, then $3a \in \mathbb{Z}_p^*$. By Lemma 1.5, it is enough to consider $ax_1 + ax_2 + ax_3 = 0$. Furthermore, because $a \in \mathbb{Z}_p^*$, the triple (s_1, s_2, s_3) is a solution to $ax_1 + ax_2 + ax_3 = 0$ if and only if it is a solution to $x_1 + x_2 + x_3 = 0$. Thus, without loss of generality, the rest of the argument only considers $x_1 + x_2 + x_3 = 0$. The exact rainbow-free 3-coloring of $\mathbb{Z}_p$ with color classes $\{0\}, \{1, p-1\}, \{2, 3, \ldots, p-2\}$ establishes that $4 \leq \mathrm{rb}(\mathbb{Z}_p, \mathrm{eq}(1, 1, 1, 0))$. Suppose there is an exact 4-coloring of $\mathbb{Z}_p$ with color classes A, B, C, and D such that $|A| \leq |B| \leq |C| \leq |D|$. It will be shown that a rainbow solution exists in the aforementioned exact 4-coloring.

Case 1: At most one color class has size one.

Consider the exact 3-colorings with color classes: $A \cup B$, C, D; B, $A \cup C$, D; and B, C, $A \cup D$. If each of the colorings is rainbow-free, then, by Theorem 1.4, each of the three colorings are arithmetic progressions. In particular, $A \cup B$, C and D are arithmetic progressions with common difference d; $A \cup C$, B and D are arithmetic progressions with common difference d'; and $A \cup D$, B and C are arithmetic progressions with common difference d''. Since the first two partitions overlap in the set D, and $2 \leq |D| \leq p - 2$, we know that $d' = \pm d$ by Lemma 2.7. Similarly, $d'' = \pm d$. Without loss of generality, $d'' = d' = \pm d$. However, any arithmetic progression with common difference d is also an arithmetic progression with common difference $-d$. This gives $A \cup B$, $A \cup C$, $A \cup D$, B, C, and D are all arithmetic progressions with the same common difference d. This contradicts Lemma 2.4 so one of the exact 3-colorings must have a rainbow solution.

Case 2: Exactly two color classes have size one.

Let $A = \{s\}$ and $B = \{\beta\}$. If $\beta \neq -2s$, then $\{s, \beta, -s-\beta\}$ is a rainbow solution. Thus, without loss of generality, assume $\beta = -2s$. Note this also means $s \neq 0$. Consider the exact 3-coloring with color classes $A \cup B$, C, D. If this coloring is rainbow-free, then, by Theorem 1.4.2, $A \cup B$, C and D must be arithmetic progressions with common difference d. Further, $d^{-1}(A \cup B)$, $d^{-1}C$ and $d^{-1}D$ are sets of consecutive integers and is a rainbow-free exact 3-coloring. Now consider the exact 3-coloring with color classes $d^{-1}A$, $d^{-1}(B \cup C)$, $d^{-1}D$. Theorem 1.4.1 implies that $d^{-1}(B \cup C)\backslash\{d^{-1}(-2s)\} + s2^{-1} = d^{-1}C + s2^{-1}$ and $d^{-1}D + s2^{-1}$ are symmetric. However, the color classes must also be consecutive which would imply $\beta = -s$, which is a contradiction.

Case 3: At least three color classes have size one.

Without loss of generality, dilate the coloring so that $A = \{1\}$, $B = \{\beta\}$, and $C = \{\gamma\}$. Note that if the exact 3-colorings with color classes A, C, $B \cup D$ and A, B, $C \cup D$ are rainbow-free, then they must be of the form described in Theorem 1.1 part 2.b. This means $B\backslash\{-2\} + 2^{-1} \in \{\emptyset, \{0\}\}$ and $C\backslash\{-2\} + 2^{-1} \in \{\emptyset, \{0\}\}$. So, without loss of generality, $\beta = -2$ and $\gamma = -(2^{-1})$. Notice that $(-2, -(2^{-1}), 2 + 2^{-1})$ is a rainbow solution because $-2 = -(2^{-1})$, $2 + 2^{-1} = -2$ or $2 + 2^{-1} = -(2^{-1})$ imply $p \in \{2, 3\}$.

In all cases, an exact 3-coloring constructed from the original exact 4-coloring has a rainbow solution. Thus the original exact 4-coloring has a rainbow solution. □

3 Rainbow Numbers of $\mathbb{Z}_n$ for $a_1x_1 + a_2x_2 + a_3x_3 = b$

In this section the rainbow number for $\mathbb{Z}_n$ will be established under certain conditions on the coefficients. Since 2 is a special case, the rainbow number for $\mathbb{Z}_{2^\alpha}$ will be considered first. Then the lower and upper bounds are established for general n.

Let A and B be sets and $m, n \in \mathbb{Z}$. (A, m, eq) is *solomorphic* to (B, n, eq') when there exists a function $\phi : A \to B$ such that $\{s_1, s_2, s_3\} \subset A$ is a solution to eq mod m if and only if $\{\phi(s_1), \phi(s_2), \phi(s_3)\} \subset B$ is a solution to eq' mod n. Note that solomorphic sets have the same rainbow number.

Theorem 3.1. *If $a_1a_2a_3 \in \mathbb{Z}_2^*$, then*

$$\text{rb}(\mathbb{Z}_{2^\alpha}, \text{eq}(a_1, a_2, a_3, b)) = \alpha + 2.$$

Proof. The proof follows by induction on α. The base case $\alpha = 1$ holds by convention. Note that since $a_1a_2a_3 \in \mathbb{Z}_2^*$, then $a_i \equiv 1 \bmod 2$ for all i and, by Lemma 1.5, it can be assumed that $b = 0$. Let $0 \leq \alpha \in \mathbb{Z}$ and assume the statement holds for $\alpha \geq 1$. The following cases show the statement is true for $\alpha + 1$. Let c be an exact $\alpha + 2$ coloring of $\mathbb{Z}_{2^\alpha}$ and define $R_i = \{x \in \mathbb{Z}_{2^\alpha} \mid x \equiv i \bmod 2\}$ and $P_i = \{c(x) \mid x \in R_i\}$.

If at least $\alpha + 1$ colors appear in P_0, then, by the inductive hypothesis, $\mathbb{Z}_{2^\alpha}$ contains a rainbow solution to $eq = \text{eq}(a_1, a_2, a_3, 0)$ because $(\mathbb{Z}_{2^{\alpha-1}}, 2^{\alpha-1}, eq)$ is solomorphic to $(R_0, 2^\alpha, eq)$. If at most α colors appear in P_0, then there exist two colors, red and blue, that appear in P_1. Let s_1 and s_2 be two elements in R_1 such that $c(\{s_1, s_2\}) = \{red, blue\}$. Since $s_1 \equiv s_2 \equiv 1 \bmod 2$, there exists $s_3 \in \mathbb{Z}_{2^\alpha}$ such that $a_1(s_1 + s_2) + a_1s_3 \equiv 0 \bmod 2$, which implies $s_3 \in R_0$. This means $\{s_1, s_2, s_3\}$ is a rainbow solution to $\text{eq}(a_1, a_2, a_3, 0)$ since $c(s_3) \notin \{red, blue\}$. Thus $\text{rb}(\mathbb{Z}_{2^\alpha}, \text{eq}(a_1, a_2, a_3, 0)) \leq \alpha + 2$. To obtain a lower bound, color P_0 with a rainbow-free coloring of $\mathbb{Z}_{2^{\alpha-1}}$ that has α colors and color P_1 with the $(\alpha+1)^{st}$ color. This coloring has no rainbow solutions since every solution has exactly 1 or 3 elements from R_0. □

Theorem 3.2 and Corollary 3.3 establish the lower bound for the rainbow number of $\mathbb{Z}_n$.

Theorem 3.2. *Let $2 \leq t \in \mathbb{Z}$. If $a_1a_2a_3 \in \mathbb{Z}_p^*$ and $\mathbb{Z}_t$ has a rainbow-free, exact r_t-coloring for $eq = \text{eq}(a_1, a_2, a_3, 0))$ where 0 is uniquely colored, then there exists a rainbow-free, exact $(\text{rb}(\mathbb{Z}_p, eq) + r_t - 2)$-coloring of $\mathbb{Z}_{pt}$ with 0 uniquely colored.*

Proof. Since $\text{rb}(\mathbb{Z}_p, eq) \leq 4$, every rainbow-free coloring of $\mathbb{Z}_p$ for eq uses at most three colors. Define $r_p = \text{rb}(\mathbb{Z}_p, eq) - 1$. Note there exists an exact rainbow-free r_p-coloring of $\mathbb{Z}_p$ for eq where 0 is the only element in its color class. If $r_p = 2$ or $p = 3$, the r_p-coloring is obvious. If $r_p = 3$, $p \geq 5$, and $a_i \neq a_j$, for some $i \neq j$, such a coloring exists by Theorem 1.2. Lastly, if $a_1 = a_2 = a_3$ the coloring is described in the proof of Theorem 2.8.

Let c_p be a rainbow-free, exact r_p-coloring of $\mathbb{Z}_p$ for eq such that 0 is colored uniquely and c_t be a rainbow-free exact r_t-coloring of $\mathbb{Z}_t$ where 0 is colored uniquely. Define an exact $(r_p + r_t - 1)$-coloring of $\mathbb{Z}_{pt}$ by

$$c(x) = \begin{cases} 0 & \text{if } x = 0, \\ c_p(x \bmod p) & \text{if } x \neq 0 \bmod p, \\ (r_p - 1) + c_t\left(\frac{x}{p} \bmod t\right) & \text{if } x = 0 \bmod p \text{ and } x \neq 0. \end{cases}$$

Notice that $0 \in \mathbb{Z}_{pt}$ is the only element in it's color class with respect to the coloring c. Let (s_1, s_2, s_3) be a solution in $\mathbb{Z}_{pt}$ to eq. Since $a_1a_2a_3 \in \mathbb{Z}_p^*$, p cannot divide exactly two of s_1, s_2, and s_3, so either p divides each of s_1, s_2, and s_3 or p divides at most one of s_1, s_2, and s_3.

If p divides each of s_1, s_2, and s_3, then (s_1, s_2, s_3) is not a rainbow solution under the coloring c since it is not a rainbow solution under c_t. If p divides at most one of s_1, s_2, and s_3, then (s_1, s_2, s_3) is not a rainbow solution under the coloring c since it is not a rainbow solution under c_p and 0 is the unique element in its color class under c_p. Therefore, c is a rainbow-free, exact $(\mathrm{rb}(\mathbb{Z}_p, eq) + r_t - 2)$-coloring of $\mathbb{Z}_{pt}$. □

Corollary 3.3. *If $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_\ell^{\alpha_\ell}$, p_x prime for $1 \leq x \leq \ell$, and $a_1a_2a_3 \in \mathbb{Z}_n^*$, then*

$$2 + \sum_{i=1}^{\ell} [\alpha_i(\mathrm{rb}(\mathbb{Z}_{p_i}, \mathrm{eq}(a_1, a_2, a_3, 0) - 2)] \leq rb(\mathbb{Z}_n, \mathrm{eq}(a_1, a_2, a_3, 0)).$$

Proof. Define $eq = \mathrm{eq}(a_1, a_2, a_3, 0)$. This proof is inductive on the sum of the exponents in the prime factorization of n. The statement is straightforward to show when n is prime. Assume that for $k = \frac{n}{p_j} = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_j^{\alpha_j - 1} \ldots p_\ell^{\alpha_\ell} < n$, there is a rainbow-free, exact r_k-coloring of $\mathbb{Z}_k$ such that 0 is colored uniquely with $r_k = 1 + (\alpha_j - 1)(\mathrm{rb}(\mathbb{Z}_{p_j}, eq) - 2) + \sum_{\substack{i=1 \\ i \neq j}}^{\ell} \alpha_i \left[\mathrm{rb}(\mathbb{Z}_{p_i}, eq) - 2\right]$. Applying Theorem 3.2 gives a rainbow-free, exact $(\mathrm{rb}(\mathbb{Z}_{p_j}, eq) + r_k - 2)$-coloring of $\mathbb{Z}_{kp_j}$ where 0 is colored uniquely. Therefore,

$$\begin{aligned} \mathrm{rb}(\mathbb{Z}_n, eq) &\geq \mathrm{rb}(\mathbb{Z}_{p_j}, eq) + r_k - 1 \\ &\geq \mathrm{rb}(\mathbb{Z}_{p_j}, eq) + (\alpha_j - 1)(\mathrm{rb}(\mathbb{Z}_{p_j}, eq) - 2) + \sum_{\substack{i=1 \\ i \neq j}}^{\ell} \alpha_i \left[\mathrm{rb}(\mathbb{Z}_{p_i}, eq) - 2\right] \\ &= 2 + \sum_{i=1}^{\ell} \alpha_i \left[\mathrm{rb}(\mathbb{Z}_{p_i}, eq) - 2\right]. \end{aligned}$$

□

Corollary 3.4 generalizes Corollary 3.3 to $\mathrm{eq}(a_1, a_2, a_3, b)$ using Lemma 1.5.

Corollary 3.4. *If $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_\ell^{\alpha_\ell}$, p_k prime for $1 \leq k \leq \ell$, and $a_1 + a_2 + a_3, a_1a_2a_3 \in \mathbb{Z}_n^*$, then*

$$2 + \sum_{i=1}^{\ell} [\alpha_i(\mathrm{rb}(\mathbb{Z}_{p_i}, \mathrm{eq}(a_1, a_2, a_3, b) - 2)] \leq \mathrm{rb}(\mathbb{Z}_n, \mathrm{eq}(a_1, a_2, a_3, b)).$$

The upper bound will now be established. Suppose c is a coloring of $\mathbb{Z}_{ut}$. The remainder of this section uses residue classes $R_i = \{z \in \mathbb{Z}_{ut} \mid z \equiv i \bmod u\}$ and color palettes $P_i = \{c(z) \mid z \in R_i\}$ that were mentioned in the proof of Theorem 3.1.

Lemma 3.5. *Let $3 \leq t, u \in \mathbb{Z}$, $a_3 \in \mathbb{Z}_u^*$, and (s_1, s_2, s_3) and (s_1', s_2', s_3') be solutions in $\mathbb{Z}_{ut}$ to* $\mathrm{eq}(a_1, a_2, a_3, b)$*. If $s_1' \in R_{s_1}$ and $s_2' \in R_{s_2}$, then $s_3' \in R_{s_3}$.*

Proof. Since $a_1 s_1' + a_2 s_2' + a_3 s_3' = b \bmod ut$ implies $a_1 s_1' + a_2 s_2' + a_3 s_3' = b \bmod u$, solving for s_3' over $\mathbb{Z}_u$ gives

$$s_3' = a_3^{-1}(b - (a_1 s_1 + a_2 s_2)) = a_3^{-1}(a_3 s_3) \bmod u.$$

Hence, $s_3' \in R_{s_3}$. □

A similar argument to the one used in Lemma 3.5 can be used for $a_1, a_2 \in \mathbb{Z}_t^*$ and solving for s_1' and s_2' instead.

Lemma 3.6. *If $k, n \in \mathbb{Z}$ such that $3 \leq n$, then*

$$\mathrm{rb}(\mathbb{Z}_n, \mathrm{eq}(a_1, a_2, a_3, b)) = \mathrm{rb}(\mathbb{Z}_n, \mathrm{eq}(a_1, a_2, a_3, b + (a_1 + a_2 + a_3)k)).$$

Proof. Let $eq = \mathrm{eq}(a_1, a_2, a_3, b)$, $a = a_1 + a_2 + a_3$, $eq' = \mathrm{eq}(a_1, a_2, a_3, b + ak)$ and c be an exact r-coloring of $\mathbb{Z}_n$ for eq. If (s_1, s_2, s_3) is a solution in $\mathbb{Z}_n$ to eq, then $a_1 s_1 + a_2 s_2 + a_3 s_3 + (a_1 + a_2 + a_3)k = b + ak$ and $(s_1 + k, s_2 + k, s_3 + k)$ is a solution in $\mathbb{Z}_n$ to eq'. Define $c_k : \mathbb{Z}_n \to [r]$ by $c_k(x) = c(x + k \bmod n)$. Thus, (s_1, s_2, s_3) is a rainbow solution to eq with respect to c if and only if $(s_1 + k, s_2 + k, s_3 + k)$ is a rainbow solution to eq' with respect to c_k. Since c_k is a translation of the coloring c, $\mathrm{rb}(\mathbb{Z}_n, eq) = \mathrm{rb}(\mathbb{Z}_n, eq')$. □

Lemma 3.7. *Let $2 \leq t \in \mathbb{Z}$ and c be a rainbow-free coloring of $\mathbb{Z}_{ut}$ for* $\mathrm{eq}(a_1, a_2, a_3, b)$ *and $a_1 a_2 a_3 \in \mathbb{Z}_{ut}^*$ that does not use color yellow. If there exists $j \in \mathbb{Z}_t$ such that for all $i \in \mathbb{Z}_t$, $|P_i \backslash P_j| \leq 1$, then the coloring of $\mathbb{Z}_t$ given by*

$$\hat{c}(i) = \begin{cases} \textit{yellow} & P_i \subseteq P_j, \\ P_i \setminus P_j & \textit{otherwise}, \end{cases}$$

is well-defined and rainbow-free.

Proof. Since $|P_i \backslash P_j| \leq 1$, $\hat{c}$ is well-defined. Let $eq = \mathrm{eq}(a_1, a_2, a_3, b)$ and assume that (s_1, s_2, s_3) is a rainbow solution of eq in $\mathbb{Z}_t$ with respect to $\hat{c}$. Since (s_1, s_2, s_3) is a rainbow solution, without loss of generality, $\hat{c}(s_1) = \textit{red}$ and $\hat{c}(s_2) = \textit{blue}$. Thus, there exist $\alpha \in R_{s_1}$, $\delta \in R_{s_2}$ such that $c(\alpha) = \textit{red}$ and $c(\delta) = \textit{blue}$. Therefore, (α, δ, γ) is a solution to eq in $\mathbb{Z}_{ut}$ for some $\gamma \in R_{s_3}$. Note that $\hat{c}(s_3)$ is not *red* or *blue*. However, $P_{s_3} \setminus \{\hat{c}(s_3)\} \subseteq P_j$. Therefore $c(\gamma)$ is not *red* or *blue* so (α, δ, γ) is a rainbow solution to eq in $\mathbb{Z}_{ut}$ with respect to c, a contradiction. □

Lemma 3.8. *If c is a rainbow-free coloring of $\mathbb{Z}_{ut}$ for* $\mathrm{eq}(a_1, a_2, a_3, b)$*, $a_1 a_2 a_3 \in \mathbb{Z}_{ut}^*$ and $|P_0| \geq |P_i|$ for $0 \leq i \leq u - 1$, then $|P_i \backslash P_0| \leq 1$.*

Proof. Assume $|P_i \backslash P_0| \geq 2$ for some $1 \leq i \leq u - 1$ and let $\textit{red}, \textit{blue} \in P_i \backslash P_0$. Let $j \in \mathbb{Z}_{ut}$ be such that $a_1 i + a_2 0 + a_3 j = b$. Suppose there is an $\alpha \in R_j$ such that $c(\alpha) \notin P_0$. Choose $\beta \in R_i$ such that $c(\beta) \in \{\textit{red}, \textit{blue}\} \backslash \{c(\alpha)\}$. Now there exists $\gamma \in R_0$ such that $\{\beta, \gamma, \alpha\}$ is a rainbow solution to $\mathrm{eq}(a_1, a_2, a_3, b)$, a contradiction. Therefore, $P_j \subseteq P_0$. A similar argument gives that $P_0 \subseteq P_j$, so $P_0 = P_j$.

Since $|P_0|$ is maximum there must exist two colors, both in P_0 and P_j, that are not in P_i. Let $\textit{yellow}, \textit{green} \in P_0 \backslash P_i$. Choosing a *yellow* element in R_0 and a *green* element in R_j and solving for the appropriate element in R_i will give a rainbow solution, which is a contradiction. Therefore, $|P_i \backslash P_0| \leq 1$ for all $0 \leq i \leq u - 1$. □

Using an inductive argument with the following Lemma 3.9, similar to the argument made in Corollary 3.3, and Theorem 3.1 gives Corollary 3.12.

Lemma 3.9. *If $2 \leq t \in \mathbb{Z}$, $3 \leq p$ prime, $a_1 a_2 a_3 \in \mathbb{Z}_{pt}^*$, then*

$$\mathrm{rb}(\mathbb{Z}_{pt}, \mathrm{eq}(a_1, a_2, a_3, b)) \leq \mathrm{rb}(\mathbb{Z}_p, \mathrm{eq}(a_1, a_2, a_3, b_2)) + \mathrm{rb}(\mathbb{Z}_t, \mathrm{eq}(a_1, a_2, a_3, b_1)) - 2,$$

for some $b_1, b_2 \in \mathbb{Z}$.

Proof. Let $eq = \mathrm{eq}(a_1, a_2, a_3, b)$ and c be a rainbow-free exact $(\mathrm{rb}(\mathbb{Z}_{pt}, \mathrm{eq}(a_1, a_2, a_3, b)) - 1)$-coloring of $\mathbb{Z}_{pt}$. Create the coloring c_k from Lemma 3.6 to get $|P_0| \geq |P_i|$ for $1 \leq i \leq p-1$, where P_i are defined with respect to coloring c_k. This implies that $(\mathbb{Z}_{pt}, pt, eq)$ is solomorphic to $(\mathbb{Z}_{pt}, pt, eq_1)$ with $eq_1 = \mathrm{eq}(a_1, a_2, a_3, b_1)$ for some $b_1 \in \mathbb{Z}_{pt}$.

Since c_k is rainbow-free and $|P_0| \geq |P_i|$ for all i, Lemma 3.7 and Lemma 3.8 give a well-defined coloring $\hat{c}$ using P_0. If $\hat{c}$ has a rainbow solution, then c_k has a rainbow solution, so $\hat{c}$ must be rainbow-free. However, since $\hat{c}$ is coloring $\mathbb{Z}_t$, $\hat{c}$ uses at most $\mathrm{rb}(\mathbb{Z}_t, eq_1) - 1$ colors which contributes at most $\mathrm{rb}(\mathbb{Z}_t, eq_1) - 2$ colors to c because, without loss of generality, *yellow* is not a color from c. Furthermore, (R_0, pt, eq_1) is solomorphic to $(\mathbb{Z}_p, p, eq_2)$ so $|P_0| \leq \mathrm{rb}(\mathbb{Z}_p, eq_2) - 1$, where $eq_2 = \mathrm{eq}(a_1, a_2, a_3, b_2)$ for some $b_2 \in \mathbb{Z}_p$. In order for $\hat{c}$ to be rainbow-free, c_k must use at most $\mathrm{rb}(\mathbb{Z}_p, eq_2) + \mathrm{rb}(\mathbb{Z}_t, eq_1) - 3$ colors. This implies $\mathrm{rb}(\mathbb{Z}_{pt}, \mathrm{eq}(a_1, a_2, a_3, b)) - 1 \leq \mathrm{rb}(\mathbb{Z}_p, eq_2) + \mathrm{rb}(\mathbb{Z}_t, eq_1) - 3$. □

Corollary 3.10. *If $n = p_1 p_2 \cdots p_\ell$, $3 \leq p_k$ prime for $1 \leq k \leq \ell$, $a_1 a_2 a_3 \in \mathbb{Z}_n^*$, and $eq = \mathrm{eq}(a_1, a_2, a_3, b)$, then*

$$\mathrm{rb}(\mathbb{Z}_n, eq) \leq 2 + \sum_{k=1}^{\ell} \left[\mathrm{rb}(\mathbb{Z}_{p_k}, eq_k) - 2)\right],$$

where $eq_k = \mathrm{eq}(a_1, a_2, a_3, b_k)$ for some $b_k \in \mathbb{Z}$.

Corollary 3.11. *Let $n = p_1 p_2 \cdots p_\ell$, p_k prime for $1 \leq k \leq \ell$, $a_1 a_2 a_3 \in \mathbb{Z}_n^*$, where $a_1 + a_2 + a_3 \in \mathbb{Z}_3^*$ if $3 \mid n$. Let $eq = \mathrm{eq}(a_1, a_2, a_3, 0)$, then*

$$\mathrm{rb}(\mathbb{Z}_n, eq) \leq 2 + \sum_{k=1}^{\ell} \left[\mathrm{rb}(\mathbb{Z}_{p_k}, eq) - 2)\right].$$

Proof. By Theorems 2.3, 2.8, and 3.1, if $p \neq 3$, then $\mathrm{rb}(\mathbb{Z}_p, \mathrm{eq}(a_1, a_2, a_3, b)) \leq \mathrm{rb}(\mathbb{Z}_p, \mathrm{eq}(a_1, a_2, a_3, 0))$ for all b. If $a_1 + a_2 + a_3 \in \mathbb{Z}_3^*$, Proposition 2.2 gives $\mathrm{rb}(\mathbb{Z}_3, \mathrm{eq}(a_1, a_2, a_3, b)) \leq \mathrm{rb}(\mathbb{Z}_3, \mathrm{eq}(a_1, a_2, a_3, 0))$ for all b. The result follows by Corollary 3.10. □

Note that the assumption $a_1 + a_2 + a_3 \in \mathbb{Z}_3^*$ is necessary when $3 \mid n$. For example, $\mathrm{rb}(\mathbb{Z}_3, \mathrm{eq}(1, 1, 1, 0)) = 3$ and $\mathrm{rb}(\mathbb{Z}_9, \mathrm{eq}(1, 1, 1, 0)) = 5$. In particular, $\mathbb{Z}_9$ has the rainbow-free coloring $c : \mathbb{Z}_9 \to [4]$ given by $c(2) = 2$, $c(5) = 3$, $c(8) = 4$, and $c(x) = 1$ else.

Corollary 3.11 and Lemma 1.5 combine to give Corollary 3.12.

Corollary 3.12. *If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\ell^{\alpha_\ell}$, p_k prime for $1 \leq k \leq \ell$, $a_1 + a_2 + a_3, a_1 a_2 a_3 \in \mathbb{Z}_n^*$ and $eq = \mathrm{eq}(a_1, a_2, a_3, b)$, then*

$$\mathrm{rb}(\mathbb{Z}_n, eq) \leq 2 + \sum_{k=1}^{\ell} \left[\alpha_k(\mathrm{rb}(\mathbb{Z}_{p_k}, eq) - 2)\right].$$

Finally, Corollaries 3.3, 3.4 and 3.12 combine to give Theorem 3.13.

Theorem 3.13. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\ell^{\alpha_\ell}$, with p_k prime for $1 \leq k \leq \ell$, and $a_1 a_2 a_3 \in \mathbb{Z}_n^*$. If one of the following holds:*

1) $b \neq 0$ and $a_1 + a_2 + a_3 \in \mathbb{Z}_n^*$,

2) $b = 0$ and $3 \nmid n$, or

3) $b = 0$, $3 \mid n$, and $a_1 + a_2 + a_3 \in \mathbb{Z}_3^*$,

then

$$\mathrm{rb}(\mathbb{Z}_n, \mathrm{eq}(a_1, a_2, a_3, b)) = 2 + \sum_{k=1}^{\ell} \left[\alpha_k(\mathrm{rb}(\mathbb{Z}_{p_k}, \mathrm{eq}(a_1, a_2, a_3, b) - 2)\right].$$

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