Polychromatic Colorings on the Integers

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Abstract

We show that for any set $S \subseteq \mathbb{Z}$, $|S| = 4$ there exists a 3-coloring of $\mathbb{Z}$ in which every translate of S receives all three colors. This implies that S has a codensity of at most 1/3, proving a conjecture of Newman [D. J. Newman, Complements of finite sets of integers, *Michigan Math. J.* 14 (1967) 481–486]. We also consider related questions in $\mathbb{Z}^d$, $d \geq 2$.

1 Introduction

Throughout the paper, let G denote an arbitrary abelian group. Given $S, T \subseteq G$, $n \in G$, define $S + T = \{s + t : s \in S, t \in T\}$ and $n + S = \{n\} + S$. Any set of the form $n + S$ is called a *translate* of S. Given a subset S of G, a coloring of the elements of G is S*-polychromatic* if every translate of S contains an element of each color. Define the *polychromatic number* of S, denoted $p_G(S)$, to be the largest number of colors allowing an S-polychromatic coloring of the elements of G. We just write $p(S)$ when the choice of G is clear from context.

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We begin with some elementary observations that will be used repeatedly. First, if S' is a subset of S, then any S'-polychromatic coloring is also an S-polychromatic coloring, and thus $p(S') \leq p(S)$. Also, if $S' = n + S$ is a translate of S, then S' and S have the same set of translates, so $p(S') = p(S)$.

We are primarily concerned with the setting where $G = \mathbb{Z}$ and S is finite. If $S \subseteq \mathbb{Z}$ has cardinality 1 or 2, $p(S) = |S|$. For $|S| = 3$, $p(S)$ can be 2 or 3. For example, if $S = \{0, 1, 5\}$ then every translate of S contains three elements which are each in different congruence classes $\pmod 3$. Thus a 3-coloring of the integers where each congruence class (mod 3) is colored with a different color is S-polychromatic, and $p(\{0, 1, 5\}) = 3$. However $p(\{0, 1, 3\}) = 2$. To see that $p(\{0, 1, 3\}) \neq 3$, let χ be a 3-coloring of $\mathbb{Z}$ with $\chi(0)$, $\chi(1)$, and $\chi(3)$ all different. Some element $s \in \{0, 1, 3\}$ has $\chi(s) = \chi(2)$, and there is a translate of $\{0, 1, 3\}$ that contains both s and 2, so the coloring is not polychromatic. Our main result concerns the polychromatic numbers of sets with cardinality 4.

Theorem 1 *If $S \subseteq \mathbb{Z}$ and $|S| = 4$, then $p(S) \geq 3$.*

The proof of Theorem 1 is given in Section 2. For larger sets S, Alon, Kříž, and Nešetřil [2] proved that $p(S) \geq \frac{(1+o(1))|S|}{3 \ln |S|}$, while there exists some set S where $p(S) \leq \frac{(1+o(1))|S|}{\ln |S|}$. Subsequently, Harris and Srinivasan [6] established a tight asymptotic lower bound on polychromatic numbers.

Theorem 2 ([2], [6]) *For a finite set $S \subseteq \mathbb{Z}$, $p(S) \geq \frac{(1+o(1))|S|}{\ln |S|}$. Moreover, there exists some set S where $p(S) \leq \frac{(1+o(1))|S|}{\ln |S|}$.*

One motivation for studying polychromatic numbers is that they provide bounds for Turán type problems (see for example [1], [10], [11]). Call $T \subseteq G$ a *blocking set* for S if $G \setminus T$ contains no translate of S, i.e. if for all $n \in G$, $n + S \nsubseteq G \setminus T$. A Turán type problem asks for the smallest blocking set for a given set S. In the case where S is finite and $G = \mathbb{Z}$, any blocking set is countably infinite, so we ask how small the density of a blocking set can be. Following the notation of Newman [8], (he worked in the setting of the natural numbers, but the definitions are equivalent), define for any set $T \subseteq \mathbb{Z}$ its *upper density* $\overline{d}(T)$ and *lower density* $\underline{d}(T)$ as

$$\overline{d}(T) = \limsup_{n\to\infty} \frac{|T \cap [-n, n]|}{2n+1} \qquad \text{and} \qquad \underline{d}(T) = \liminf_{n\to\infty} \frac{|T \cap [-n, n]|}{2n+1}.$$

If $\overline{d}(T) = \underline{d}(T)$, we call this quantity the *density* of T and denote it by $d(T)$. Define $\alpha(S)$ to be a measure of how small the density of a blocking set for S can be. Let

$$\alpha(S) = \inf\{d(T) : \ T \text{ is a blocking set for } S \text{ and } d(T) \text{ exists}\}.$$

In Section 3, we describe the relationship between polychromatic colorings and blocking sets, and prove Lemma 3.

Lemma 3 *For any finite set $S \subseteq \mathbb{Z}$, $\alpha(S) \leq 1/p(S)$.*

One of the main consequences of Theorem 1 concerns covering densities of sets of integers. Given a set $S \subseteq G$, we say $T \subseteq G$ is a *complement set* for S if $S + T = G$. We say S *tiles* G *by translation* if it has a complement set T such that if $s_1, s_2 \in S$, $t_1, t_2 \in T$, then $s_1 + t_1 = s_2 + t_2$ implies $s_1 = s_2$ and $t_1 = t_2$. We call such a complement set T a *tiling complement set* for S. Note that the set S tiles G by translation if all the translates $S + t$, $t \in T$, are disjoint and every $n \in G$ is an element of some translate $S + t$. In this paper we only consider tilings by translation, so if S tiles G by translation with tiling complement set T we will simply say S tiles G and write $G = S \oplus T$.

Again, our primary interest will be the case where $G = \mathbb{Z}$ and S is finite. For example, if $S = \{0, 1, 5\}$, then S tiles $\mathbb{Z}$ with complement set $T = \{3n : n \in \mathbb{Z}\}$. However $S = \{0, 1, 3\}$ does not tile $\mathbb{Z}$. Newman [9] proved necessary and sufficient conditions for a finite set S to tile $\mathbb{Z}$ if $|S|$ is a power of a prime.

Theorem 4 (Newman [9]) *Let $S = \{s_1, \ldots, s_k\}$ be distinct integers with $|S| = p^\alpha$ where p is prime and α is a positive integer. For $1 \leq i < j \leq k$ let $p^{e_{ij}}$ be the highest power of p that divides $s_i - s_j$. Then S tiles $\mathbb{Z}$ if and only if $|\{e_{ij} : 1 \leq i < j \leq k\}| \leq \alpha$.*

Later Coven and Meyerowitz [5] gave necessary and sufficient conditions for S to tile $\mathbb{Z}$ when $|S| = p_1^{\alpha_1} p_2^{\alpha_2}$, where p_1 and p_2 are primes. The general question is still open. Kolountzakis and Matolcsi [7] and Amiot [3] have published recent work motivated by what are called rhythmic tilings in music.

If a finite set S tiles $\mathbb{Z}$, it has a complement set of density $1/|S|$. Following Newman [8], we define the *codensity* of a set S, denoted $c(S)$, as a measure of how small the density of a complement set can be. Let

$$c(S) = \inf\{d(T) : S + T = \mathbb{Z} \text{ and } d(T) \text{ exists}\}.$$

We are interested in the largest codensities for sets of a given cardinality. Define

$$c_k = \sup_{\{S:|S|=k\}} c(S).$$

An example of a complement set for $\{0, 1, 3\}$ is $\{t \in \mathbb{Z} : t \equiv 0 \text{ or } 1 \pmod 5\}$, so $c(\{0, 1, 3\}) \leq 2/5$. The following theorem and conjecture on c_4 are due to Newman.

Theorem 5 (Newman [8])

- $c(\{0, 1, 3\}) = 2/5$.
- $c_3 = 2/5$.
- $c(\{0, 1, 2, 4\}) = 1/3$.

Conjecture 6 (Newman [8]) $c_4 = 1/3$.

Conjecture 6 is stated and attributed to Newman by Weinstein [16], who proved that $c_4 < .339934$. Based on a computer search, Bollobás, Janson, and Riordan [4] confirmed Newman's conjecture for sets with diameter at most 22, where the *diameter* of a nonempty finite set of integers is defined to be the difference between the largest and smallest elements in the set. They also conjectured that $c_5 = 3/11$ and $c_6 = 1/4$ (See Remark 5.6 and Question 5.7 in [4]. Note they use different notation).

In Section 3 we prove the following lemma relating blocking sets and complement sets.

Lemma 7 *For any finite set $S \subseteq \mathbb{Z}$, $c(S) = \alpha(S)$.*

Theorem 1, along with Lemmas 3 and 7, suffice to resolve Conjecture 6.

Theorem 8 *$c_4 = 1/3$.*

Proof: Theorem 5 implies $c(\{0, 1, 2, 4\}) = 1/3$, so it remains to show that for any other set S with cardinality four, $c(S) \leq 1/3$. Let $S \subseteq \mathbb{Z}$ have four elements. Then Theorem 1 implies that $p(S) \geq 3$, and by Lemmas 3 and 7,

$$c(S) = \alpha(S) \leq 1/p(S) \leq 1/3.$$

■

In Subsection 3.1 we consider the relationship between polychromatic colorings and tilings. The main result is Theorem 14, which states that a set S tiles an abelian group G by translation if and only if $p(S) = |S|$.

Finally, in Section 4 we turn our attention to polychromatic numbers and tilings for finite sets in $\mathbb{Z}^d$. We begin by proving in Theorem 20 that the bound of Theorem 2 applies to subsets of $\mathbb{Z}^d$. We then show that if a set of points in $\mathbb{Z}^d$ is collinear, determining its polychromatic number is equivalent to determining the polychromatic number of a specific projection of this set into $\mathbb{Z}$. Theorem 14 implies that a set S tiles $\mathbb{Z}^d$ if and only if $p_{\mathbb{Z}_d}(S) = |S|$, so we use this to restate some well-known results on tilings of $\mathbb{Z}^d$ by finite sets in the language of polychromatic colorings. We conclude by applying these results to determine polychromatic numbers of sets with cardinality 3 and 4 in $\mathbb{Z}^d$.

2 Sets of Cardinality Four

In this section we prove that every set of four integers has polychromatic number at least 3. We begin by stating some general lemmas that reduce the problem of finding an S-polychromatic coloring of $\mathbb{Z}$ to finding an S-polychromatic coloring of $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ for a specific choice of m.

Lemma 9 *If G and H are abelian groups and $\phi : G \to H$ is a homomorphism, then* *for all* $S \subseteq G$,

$$p_G(S) \geq p_H(\phi(S)).$$

Proof: Let $S \subseteq G$ and let χ' be a $\phi(S)$-polychromatic coloring of H with $p_H(\phi(S))$ colors. Define the coloring χ on G such that $\chi(g) = \chi'(\phi(g))$. Consider a translate $g + S$ of the set S. Since

$$\chi(g+S) = \chi'(\phi(g+S)) = \chi'(\phi(g) + \phi(S)),$$

and $\phi(g) + \phi(S) \subseteq H$ is a translate of $\phi(S)$, $\chi(g+S)$ contains all $p_H(\phi(S))$ colors, and χ is S-polychromatic. ■

Corollary 10 *If G and H are abelian groups and $\phi : G \to H$ is an isomorphism, then for all $S \subseteq G$,*

$$p_G(S) = p_H(\phi(S)).$$

Lemma 11 *If H is a subgroup of an abelian group G, and $S \subseteq H$, then $p_H(S) = p_G(S)$.*

Proof: Since H is a subset of G, $p_H(S) \geq p_G(S)$. To prove the other inequality, suppose χ' is an S-polychromatic coloring of H with $p_H(S)$ colors. Let $V \subseteq G$ be a set containing exactly one element from each coset of H. For every $g \in G$, there is a unique $h \in H$ and $v \in V$ such that $g = h + v$. Define a coloring χ of G such that $\chi(g) = \chi'(h)$, i.e. the summand v is ignored. We show that χ is S-polychromatic. Given $g \in G$ with $g = h + v$ for some $h \in H$, $v \in V$, consider the translate $g + S$, and note

$$\chi(g+S) = \chi(h+v+S) = \chi'(h+S).$$

Since $h + S \subseteq H$ is a translate of S, $\chi(g+S)$ contains all $p_H(\phi(S))$ colors, and χ is S-polychromatic. ■

Lemma 12 *Suppose $a, b, c, k \in \mathbb{Z}$ with $0 < a < b < c$, $k \geq 1$. Let $S = \{0, ka, kb, kc\}$, $S_1 = \{0, a, b, c\}$, and $S_2 = \{0, b-a, b, 2b-a\}$. Then*

(i) $p_{\mathbb{Z}}(S) = p_{\mathbb{Z}}(S_1)$.

(ii) If $q \in \mathbb{N}$, then $p_{\mathbb{Z}}(S_1) \geq p_{\mathbb{Z}_q}(S_1)$.

(iii) If $m = c - a + b$, then $p_{\mathbb{Z}_m}(S_1) = p_{\mathbb{Z}_m}(S_2)$.

(iv) If $q \in \mathbb{N}$, with $\gcd(k, q) = 1$, then $p_{\mathbb{Z}_q}(S) = p_{\mathbb{Z}_q}(S_1)$.

Proof:

(i) Define $\phi : \mathbb{Z} \to k\mathbb{Z}$ such that $\phi(n) = kn$. Then ϕ is an isomorphism where $\phi(S_1) = S$, so Corollary 10 implies $p_{k\mathbb{Z}}(S) = p_{\mathbb{Z}}(S_1)$. Since $k\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ and $S \subseteq k\mathbb{Z}$, Lemma 11 implies $p_{k\mathbb{Z}}(S) = p_{\mathbb{Z}}(S)$. By combining these equations, we conclude $p_{\mathbb{Z}}(S) = p_{k\mathbb{Z}}(S) = p_{\mathbb{Z}}(S_1)$.

(ii) This part follows from Lemma 9 using the homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}_q$ where $\phi(n) = n \pmod q$.

(iii) In $\mathbb{Z}_m$, with addition $\pmod m$, $S_2 = S_1 + (b-a)$. Thus in $\mathbb{Z}_m$, S_1 and S_2 are translates of each other and have the same polychromatic number.

(iv) Define $\phi : \mathbb{Z}_q \to \mathbb{Z}_q$ so that $\phi(n) = kn$. Since $\gcd(k, q) = 1$, ϕ is an isomorphism. Since $\phi(S_1) = S$, Corollary 10 implies $p_{\mathbb{Z}_q}(S) = p_{\mathbb{Z}_q}(S_1)$.

■

Proof of Theorem 1: Let $S \subseteq \mathbb{Z}$ have cardinality four. Since all translates of S have the same polychromatic number, we may assume that 0 is the smallest element of S, and by Lemma 12, Part (i), it suffices to prove the theorem in the case that $S = \{0, a, b, c\}$ with $0 < a < b < c$ and $\gcd(a, b, c) = 1$.

It is possible, though tedious, to prove the entire theorem by hand. Thus in the interest of simplifying the exposition, we verified using a computer search that for every S with diameter at most 288 there exists an S-polychromatic 3-coloring of $\mathbb{Z}_q$ for some q depending on S. The code for this search has been included as an ancillary file with the preprint of this paper at arxiv.org/abs/1704.00042. By Lemma 12, Part (ii), this gives a periodic S-polychromatic 3-coloring of $\mathbb{Z}$. Hence we suppose that $c \geq 289$.

For the remainder of the proof, let $m = c - a + b$. By Lemma 12, Parts (ii) and (iii), it suffices to show that we can 3-color $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ so that the translates of $\{0, b-a, b, 2b-a\}$ are polychromatic. So for the remainder of the proof we assume $S = \{0, b-a, b, 2b-a\}$ and seek an S-polychromatic 3-coloring of $\mathbb{Z}_m$. The key observation regarding S is that it contains two repeated differences: $b-a$ and b.

Define $d_1 = \gcd(b, m)$ and $d_2 = \gcd(b-a, m)$. Since $1 = \gcd(a, b, c) = \gcd(b - a, b, c - a + b) = \gcd(b-a, b, m)$, we know $\gcd(d_1, d_2) = 1$. We distinguish two main cases. In the first case, which we call "single cycle," we assume $\min\{d_1, d_2\} = 1$ and give a coloring of $\mathbb{Z}_m$. In the second case, which we call "multiple cycle," we assume $\min\{d_1, d_2\} > 1$ and partition $\mathbb{Z}_m$ into multiple cycles of length m/d_i for one of the choices of i. We then give a rule for coloring each cycle.

Main case 1 (Single cycle): Suppose $\min\{d_1, d_2\} = 1$. Without loss of generality, assume $d_1 = 1$ (if not, then simply switch all occurences of b and $b-a$ in the argument below). Let $2 \leq g \leq m-2$ satisfy $gb \equiv b - a \pmod m$, so that $S = \{0, bg, b, b(g+1)\}$. Applying Lemma 12, Part (iv), with $q = m$ and $k = b$, we can instead work with $S = \{0, g, 1, g+1\} = \{0, 1, g, g+1\}$.

We may assume that $g \leq m/2$, as otherwise we could work with the translate $(m - g) + S = \{0, 1, m-g, m-g+1\}$. Let s be the smallest multiple of 3 such that $g > \lceil m/s \rceil$. We consider four subcases: The first two are (1a) $g = 2$, 3, or 4 and (1b) $5 \leq g < 2\lfloor m/s \rfloor$. In the remaining subcases (1c) and (1d), $2\lfloor m/s \rfloor \leq g \leq$

$\lceil m/(s-3)\rceil$. For $m > 8$, if $2\lfloor m/s\rfloor \le g \le m/2$ then $s > 3$, and for $m > 44$, if $2\lfloor m/s\rfloor \le g \le \lceil m/(s-3)\rceil$ then $s < 9$. Since $m > c \ge 289 > 44$, we can assume $s = 6$, so $2\lfloor m/6\rfloor \le g \le \lceil m/3\rceil$. This implies $m = 3g+k$ where $-2 \le k \le 5$ and there are two further subcases to consider, depending on the residue class of m modulo 6: (1c) $m = 3g-2$, $3g-1$, $3g+1$, $3g+2$, $3g+4$, or $3g+5$, and (1d) $m = 3g$ or $3g+3$.

Subcase (1a): Suppose $g = 2$, 3, or 4. Then $S = \{0,1,2,3\}$, $\{0,1,3,4\}$, or $\{0,1,4,5\}$, respectively. In Subcase (1c) we will construct S-polychromatic 3-colorings of $\mathbb{Z}_m$ for each of these sets.

Subcase (1b): Suppose $5 \le g < 2\lfloor m/s\rfloor$. Then split $\mathbb{Z}_m$ into s intervals as equally as possible (i.e. of lengths $\lfloor m/s\rfloor$ and $\lceil m/s\rceil$) and color these intervals $010101\ldots$, followed by $121212\ldots$, then $202020\ldots$, repeating $s/3$ times. Since $\lceil m/s\rceil < g < 2\lfloor m/s\rfloor$, any translate of S' where the pairs $\{0,1\}$ and $\{g,g+1\}$ lie in different intervals gets all three colors. If one of the pairs $\{0,1\}$ or $\{g,g+1\}$ straddles two consecutive intervals, this pair may get only the single color common to these two intervals, but then the other pair lies fully inside a third interval which is colored with the remaining two colors.

Subcase (1c): Suppose $m = 3g-2$, $3g-1$, $3g+1$, $3g+2$, $3g+4$, or $3g+5$. In this case we know that $m \not\equiv 0 \pmod 3$ so we can apply Lemma 12, Part (iv), with $q = m$ and $k = 3$, and instead work with one of the sets in $\mathcal{S} = \{\{0,2,3,5\},\{0,1,3,4\},\{0,1,2,3\},\{0,3,4,7\},\{0,3,5,8\}\}$. For example, if $m = 3g-2$, then multiplying by 3, S is transformed into $\{0,3,3g,3g+3\} \equiv \{0,2,3,5\}$, while if $m = 3g+4$, then multiplying by 3, S is transformed into $\{0,3,3g,3g+3\} \equiv \{0,3,-4,-1\}$, which is a translate of $\{0,3,4,7\}$.

Thus we have reduced the problem to finding an S-polychromatic 3-coloring of $\mathbb{Z}_m$ for each of the sets $S \in \mathcal{S}$. For each $S \in \mathcal{S}$, in Table 1 we list one interval of length r and one of length $r+1$ obtained by adding an initial 0 to the other interval. We also include an interval for $\{0,1,4,5\}$ to cover Subcase (1a). Each of the intervals has the property that concatenating the intervals of length r and $r+1$ in any way results in an S-polychromatic coloring for the corresponding set. One can check this by hand, using the fact that in each case, a translate of $S \in \mathcal{S}$ intesects at most two consecutive intervals. Hence if m can be expressed as a positive integer combination of r and $r+1$, $m = hr + k(r+1)$, we can obtain an S-polychromatic coloring with period m. For $r = 3,6,7,9$, by the 2-coin Frobenius problem, m can be expressed as a positive integer combination of r and $r+1$ for any m greater than $r^2-r-1 \le 71 < 289$.

Subcase (1d): Suppose $m = 3g$ or $3g+3$. If $g \not\equiv 0 \pmod 3$ then simply color $\mathbb{Z}_m$ with the pattern $0120120\ldots012$. If $g \equiv 0 \pmod 3$ and $m = 3g$, color $\mathbb{Z}_m$ in 3 equal intervals, each of length g: $012012\ldots012$ followed by $120120\ldots120$ followed by $201201\ldots201$. Finally, if $g \equiv 0 \pmod 3$ and $m = 3g+3$ we color $\mathbb{Z}_m$ in 3 equal intervals, each of length $g+1$: $012012\ldots0120$ followed by $201201\ldots2012$ followed by $120120\ldots1201$.

S	r	period r	period $r+1$
$\{0,2,3,5\}$	6	001122	0001122
$\{0,1,3,4\}$	6	001212	0001212
$\{0,1,2,3\}$	3	012	0012
$\{0,3,4,7\}$	9	000111222	0000111222
$\{0,3,5,8\}$	9	000111222	0000111222
$\{0,1,4,5\}$	7	0001212	00001212

Table 1: One interval of a periodic coloring for sets in Subcases (1a) and (1c).

Main case 2 (Multiple cycles): Suppose $\min\{d_1, d_2\} > 1$. Since d_1 and d_2 are relatively prime, at most one of them can be a multiple of 3. Choose the smallest of these numbers that is not a multiple of 3, and as in the single cycle case, without loss of generality assume it is d_1.

Let $e_1 = m/d_1$ and $e_2 = m/d_2$. For $0 \le i < d_1$, let

$$C_i = \{(b-a)i + bj \pmod m : 0 \le j < e_1\}.$$

Since

$$\mathbb{Z}_m = \{(b-a)i + bj \pmod m : 0 \le i < d_1, 0 \le j < e_1\},$$

the C_i's form a partition of $\mathbb{Z}_m$ into d_1 cycles, each with e_1 elements.

Let $c_{i,j}$ denote the jth element of C_i, i.e. $c_{i,j} = i(b-a) + jb \pmod m$. Note that any translate of S contains two consecutive elements of two consecutive cycles, i.e. any translate of S has the form $\{c_{i,j}, c_{i,j+1}, c_{i+1,j}, c_{i+1,j+1}\}$, where the first entry in the subscript is taken $\bmod d_1$ and the second entry is taken $\bmod e_1$. We describe an S-polychromatic 3-coloring for each of four subcases: (2a) e_1 is even, (2b) d_1 is even and e_1 is odd, (2c) d_1 and e_1 are both odd, with $e_1 \le 17$, and (2d) d_1 and e_1 are both odd, with $e_1 \ge 19$.

Subcase (2a): Suppose e_1 is even. For $i = 0, \ldots, \lfloor d_1/2 \rfloor - 1$, color each C_{2i} by $01010\ldots01$ and each C_{2i+1} by $02020\ldots02$. Finally, if d_1 is odd, color C_{d_1-1} by $1212\ldots12$.

Subcase (2b): Suppose d_1 is even and e_1 is odd. For $i = 0, \ldots, d_1/2 - 1$, color each C_{2i} by $01010\ldots011$ and each C_{2i+1} by $22020\ldots02$.

Subcase (2c): Suppose d_1 and e_1 are both odd, with $e_1 \le 17$. Since $e_1 e_2 \ge m > c \ge 289$, one of e_1 and e_2 is larger than 17, so $e_2 > e_1$ and hence $d_1 > d_2$. Since d_1 is the smaller of d_1 and d_2 that is not a multiple of 3, d_2 must be a multiple of 3, and thus so is e_1.

We color each C_i with one of three patterns: $012012\ldots012$, $120120\ldots120$, or $201201\ldots201$. Such a coloring is S-polychromatic so long as for all i, C_i and C_{i+1} are colored with different patterns. For $0 \leq i \leq (d_1-3)/2$, color C_{2i} with the first pattern and color C_{2i+1} with the second pattern. Finally, color C_{d_1-1} with the third pattern.

Subcase (2d): Suppose d_1 and e_1 are both odd, with $e_1 \geq 19$. Since d_1 is not divisible by 3 and $\min\{d_1, d_2\} > 1$, $d_1 \geq 5$. Let $e_1 = u + v + w$ be a sum of odd integers u, v, w with $u \geq v \geq w \geq u-2$. Color C_0 in intervals of size u, v, w, using the patterns $0101\ldots010$ then $1212\ldots121$ and then $2020\ldots202$. For each $i \geq 1$, color C_i by taking a "counterclockwise rotation" of length r_i of the coloring of C_{i-1}, so that the color of $c_{i,j+r}$ is the same as the color of $c_{i-1,j}$. For $1 \leq i \leq d_1 - 1$, if $u \leq r_i \leq v + w = e_1 - u$, then each translate of S meeting C_{i-1} and C_i receives all 3 colors.

It remains to show that there are choices of $r_1, \ldots, r_{d_1-1}$ with $u \leq r_i \leq v+w = e_1-u$ so that of the translates of S meeting C_{d_1-1} and C_0 receive all three colors. The coloring of C_0 is a "clockwise rotation" of length $R = -r_1 - r_2 - \cdots - r_{d_1-1}$ of the coloring of C_{d_1-1}, i.e. the color of $c_{0,j-R}$ is the same as the color of $c_{d_1-1,j}$. Since for each i, $u \leq r_i \leq v + w = e_1 - u$, it suffices to show that there is a multiple of e_1 in the interval $[d_1 u, d_1(e_1 - u)]$, ensuring there are choices for the r_i's such that R is congruent to a number between u and $e_1 - u \pmod{e_1}$. This certainly holds if $d_1(e_1 - 2u) \geq e_1 - 1$ which, since $d_1 \geq 5$, holds if $4e_1 \geq 10u - 1$. This inequality is true for $e_1 \geq 19$.

This completes the multiple cycles case and the proof. ∎

3 Colorings, Blocking Sets, Coverings, and Tilings

In this section we prove the results necessary to resolve Newman's conjecture. The key insight in proving Lemma 3 is that the elements of a given color in an S-polychromatic coloring form a blocking set for S. While it is possible for $\alpha(S)$ to be equal to $1/p(S)$ (e.g. if $|S| = 2$ then $\alpha(S) = 1/2 = 1/p(S)$), in general these two quantities are not equal. For example, $p(\{0,1,3\}) = 2$, but by Lemma 7 and Theorem 5, $\alpha(\{0,1,3\}) = 2/5 < 1/2$.

Proof of Lemma 3: Let χ be an S-polychromatic coloring of $\mathbb{Z}$ with $p(S)$ colors. Suppose $d \in \mathbb{Z}$ is greater than the diameter of S and let $I_j = \{n \in \mathbb{Z} : jd \leq n < (j+1)d\}$. By the pigeonhole principle, for some $0 \leq j_1 < j_2 \leq (p(S))^d$ the coloring of the intervals I_{j_1} and I_{j_2} are identical, i.e. for $0 \leq k < d$, $\chi(j_1 d + k) = \chi(j_2 d + k)$. Let $m = (j_2 - j_1)d$. For any $n \in \mathbb{Z}$, denote by r the remainder when n is divided by m, so $0 \leq r < m$. Let χ' be the coloring of $\mathbb{Z}$ where $\chi'(n) = \chi(j_1 d + r)$. Note that χ' uses $p(S)$ colors and is periodic with period m, i.e. for all $n \in \mathbb{Z}$, $\chi(n) = \chi(n+m)$. Furthermore, the coloring under χ' of any d consecutive integers is identical to the coloring under χ of some d consecutive integers, so χ' is S-polychromatic. Let $T_i = \{n \in \mathbb{Z} : \chi'(n) = i\}$. Since any periodic set has a defined density, $d(T_i)$ is defined for each i, and $\sum_{i=1}^{p(S)} d(T_i) = 1$. Since χ' is S-polychromatic, for each i, each translate of S contains an element of T_i, i.e. T_i is also a blocking set for S. Thus for some i, T_i is a blocking set for S with density at most $1/p(S)$, which implies that $\alpha(S) \leq 1/p(S)$. ■

For any subset T of an abelian group G, let $-T$ denote the set $\{-t : t \in T\}$. Lemma 13 is well-known (see e.g. [14]) but for completeness we present a proof.

Lemma 13 *Let G be an abelian group, and $S \subseteq G$. Then $T \subseteq G$ is a complement set for S if and only if $-T$ is a blocking set for S.*

Proof: Suppose T is a complement set for S. For any $n \in G$, $-n \in S + T$, so there must be some $t \in T, s \in S$ such that $t + s = -n$. This implies $t = -n - s$, so $-n - s \in T$, and $n + s \in -T$. Thus for every n, some element of $n + S$ is in $-T$, and $-T$ is a blocking set for S.

Conversely, suppose $-T$ is a blocking set for S. For the sake of contradiction, assume T is not a complement set for S, i.e. there is some $-n \in G$ such that $-n \notin S + T$. This implies that for all $s \in S$, $-n - s \notin T$, which means for all $s \in S$, $n + s \notin -T$. Thus $n + S \subseteq G \setminus -T$, and so $-T$ is not a blocking set for S, a contradiction. ■

Proof of Lemma 7: Lemma 13 implies that T is a complement set for S if and only if $-T$ is a blocking set for S. If they exist, the densities of T and $-T$ are the same. ■

3.1 Polychromatic Colorings and Tilings

We now describe some relationships between polychromatic colorings and tilings.

Theorem 14 *Let G be any abelian group. A finite set $S \subseteq G$ tiles G by translation if and only if $p(S) = |S|$. Moreover, if χ is an S-polychromatic coloring of G with $|S|$ colors and T is the set of elements of G colored by χ with any given color, then $S \oplus T = G$.*

Proof: Let $S = \{s_1, s_2, \ldots, s_k\}$, and suppose S tiles G with complement set $T \subseteq G$. For each $n \in G$, define a coloring χ on G so that $\chi(n) = i$ if $n = s_i + t$ for some $t \in T$. By the definition of tiling, this coloring is well-defined. For the sake of contradiction, assume χ is not S-polychromatic. Then for some l where $1 \leq l \leq k$, there exists $n \in G$ and $s_i, s_j \in S$ with $i \neq j$ such that $\chi(n + s_i) = \chi(n + s_j) = l$. Then there exist $t_1, t_2 \in T$, $t_1 \neq t_2$, such that $n + s_i = t_1 + s_l$ and $n + s_j = t_2 + s_l$. Subtracting these equations, we find that $s_i - s_j = t_1 - t_2$. Thus $t_2 + s_i = t_1 + s_j$, which is a contradiction.

Conversely, let $S = \{s_1, s_2, \ldots, s_k\}$, suppose $p(S) = |S|$, and let χ be an S-polychromatic coloring of G with $|S|$ colors. Then for all $n \in G$, if $i \neq j$ then $\chi(n + s_i) \neq \chi(n + s_j)$. Let $T \subseteq G$ be the set of elements colored with a given color. We show that $S \oplus T = G$. First assume for the sake of contradiction that two translates of S share an element, i.e. there exist $s_i, s_j \in S$, $i \neq j$, $t_1, t_2 \in T$, $t_1 \neq t_2$, such that $s_i + t_1 = s_j + t_2$. Let $n = t_1 - s_j = t_2 - s_i$, so $t_1 = n + s_j$ and $t_2 = n + s_i$. Since $\chi(t_1) = \chi(t_2)$ we get $\chi(n + s_j) = \chi(n + s_i)$, so two elements of $n + S$ are colored identically, which is a contradiction.

It remains to show that $S+T = G$. Suppose there is some $n \in G$ such that $n \notin S+T$. Then for all i, $n-s_i \notin T$, which implies that the $|S|$ elements of $n-S$ are colored with at most $|S|-1$ colors, i.e. two are colored identically. Suppose $\chi(n-s_i) = \chi(n-s_j)$, where $i \neq j$. Let $m = n - s_j - s_i$. Then $m + S$ contains both $m + s_i = n - s_j$ and $m + s_j = n - s_i$. Since these integers are colored identically, $m + S$ is a translate of S that does not contain all colors, which is a contradiction. ■

Sets of integers with cardinality $n = 3$ or 4 always have polychromatic number n or $n-1$, and a corollary of Theorem 14 is that they have polychromatic number $n-1$ if and only if they do not tile $\mathbb{Z}$. According to Remark 5.6 in [4], $c(\{0,1,3,4,8\}) = 3/11 > 1/4$. Thus by Lemma 3, $\{0,1,3,4,8\}$ is an example of a set with cardinality 5 and polychromatic number 3. The results of [2] and [6] imply that for sets S with large cardinality n the cardinality and polychromatic number of S can differ by a factor of $1/\ln n$.

We now state some other corollaries of Theorem 14.

Corollary 15 *If a finite set S tiles an abelian group G by translation, then any S-polychromatic coloring of G with $|S|$ colors is also a $(-S)$-polychromatic coloring.*

Proof: Suppose S tiles G. By Theorem 14, there exists an S-polychromatic coloring χ of G with $|S|$ colors. Let $T \subseteq G$ be the set of all elements of a given color. Again by Theorem 14, $S + T = G$. Therefore by Lemma 13, $-T$ is a blocking set for S, i.e. for all $n \in G$, $n+S \nsubseteq G \setminus (-T)$. This implies that for all $n \in G$, $-n-S \nsubseteq G \setminus T$, i.e. T is a blocking set for $-S$. Since T is a blocking set for $-S$ for every color choice, every translate of $-S$ contains every color, i.e. the coloring χ is $(-S)$-polychromatic.
■

Define $t(S)$ to be the cardinality of the largest subset of S that tiles G.

Corollary 16 *For any finite subset S of an abelian group G, $p(S) \geq t(S)$.*

If $S \subseteq \mathbb{Z}$, $|S| \leq 3$, then $p(S) = t(S)$. But these parameters can be different for sets of integers with at least four elements. For example, $S = \{0,1,3,7\}$ is an example of a set where $t(S) = 2$, but $p(S) = 3$.

Question 17 *For sets S of a given cardinality, how large can the gap between $t(S)$ and $p(S)$ be?*

4 Polychromatic Colorings in $\mathbb{Z}^d$

In this section we consider polychromatic numbers in the case where $G = \mathbb{Z}^d$, $d \geq 2$. We will frequently "project" a set $S \subseteq \mathbb{Z}^d$ to another set $S' \subseteq \mathbb{Z}^{d-1}$ as follows. Let $d \geq 2$, and $\boldsymbol{w} = (w_1, \ldots, w_{d-1}, 1) \in \mathbb{Z}^d$. Define $f_{\boldsymbol{w}} : \mathbb{Z}^d \to \mathbb{Z}^{d-1}$ so that if $\boldsymbol{s} = (v_1, \ldots, v_d) \in \mathbb{Z}^d$,

$$f_{\boldsymbol{w}}(\boldsymbol{s}) = (v_1, \ldots, v_{d-1}) - v_d(w_1, \ldots, w_{d-1}).$$

We call $f_{\boldsymbol{w}}(\boldsymbol{s})$ the *projection* of $\boldsymbol{s}$ along $\boldsymbol{w}$. Given a set $S \subseteq \mathbb{Z}^d$, we call the set $f_{\boldsymbol{w}}(S) \subseteq \mathbb{Z}^{d-1}$ the *projection* of S along $\boldsymbol{w}$.

For example, if $\boldsymbol{s} = (2, 7, 4)$ and $\boldsymbol{w} = (3, 1, 1)$, the projection of $\boldsymbol{s}$ along $\boldsymbol{w}$ is $f_{\boldsymbol{w}}(\boldsymbol{s}) = (2, 7) - 4(3, 1) = (-10, 3)$. As another example, note that if $\boldsymbol{s} = (v_1, \ldots, v_d) \in \mathbb{Z}^d$, the vector $\boldsymbol{s}' = (v_1, \ldots, v_{d-1}) \in \mathbb{Z}^{d-1}$ is the projection of $\boldsymbol{s}$ along $\boldsymbol{w} = (0, \ldots, 0, 1)$.

Lemma 18 *Let $d \geq 2$, and $\boldsymbol{w} = (w_1, \ldots, w_{d-1}, 1) \in \mathbb{Z}^d$. Let $S \subseteq \mathbb{Z}^d$, and suppose $S' \subseteq \mathbb{Z}^{d-1}$ is the projection of S along $\boldsymbol{w}$. Then $p_{\mathbb{Z}^d}(S) \geq p_{\mathbb{Z}^{d-1}}(S')$.*

Proof: Since $f_{\boldsymbol{w}} : \mathbb{Z}^d \to \mathbb{Z}^{d-1}$ is a homomorphism, the result follows from Lemma 9.
■

Proposition 19 *Let $d \geq 2$. For any $S \subseteq \mathbb{Z}^d$, there is a projection $S' \subseteq \mathbb{Z}^{d-1}$ where $|S| = |S'|$.*

Proof: Let $S = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_k\} \subseteq \mathbb{Z}^d$ and suppose $\boldsymbol{w} = (w_1, \ldots, w_{d-1}, 1) \in \mathbb{Z}^d$. For $1 \leq i \leq k$ let $\boldsymbol{s}'_i = f_{\boldsymbol{w}}(\boldsymbol{s}_i)$. For $1 \leq i \leq k$, let s_{id} denote the last coordinate of $\boldsymbol{s}_i$ and note that if $i \neq j$, $\boldsymbol{s}'_i = \boldsymbol{s}'_j$ if and only if

$$\boldsymbol{w} = \frac{1}{s_{id} - s_{jd}}(\boldsymbol{s}_i - \boldsymbol{s}_j).$$

In other words $\boldsymbol{s}'_i = \boldsymbol{s}'_j$ if and only if $\boldsymbol{w}$ is parallel to $\boldsymbol{s}_i - \boldsymbol{s}_j$. Since the number of differences $\boldsymbol{s}_i - \boldsymbol{s}_j$ is finite, we can choose $\boldsymbol{w}$ so that it is not parallel to any of these. For this choice of $\boldsymbol{w}$, for all $1 \leq i \neq j \leq k$, $\boldsymbol{s}'_i \neq \boldsymbol{s}'_j$ Then $S' = \{\boldsymbol{s}'_1, \ldots, \boldsymbol{s}'_k\}$, is a projection S with $|S'| = |S|$. ■

Theorem 20 *Fix $d \geq 2$. For a finite set $S \subseteq \mathbb{Z}^d$, $p(S) \geq \frac{(1+o(1))|S|}{\ln |S|}$.*

Proof: Given $S \subseteq \mathbb{Z}^d$, Proposition 19 implies we can project $d-1$ times to ultimately obtain a set $S' \subseteq \mathbb{Z}$, with $|S'| = |S|$. Theorem 2, along with repeated application of Lemma 18, implies $p_{\mathbb{Z}^d}(S) \geq p_{\mathbb{Z}}(S') \geq \frac{(1+o(1))|S'|}{\ln |S'|} = \frac{(1+o(1))|S|}{\ln |S|}$. ■

We will be interested in the case where $S = \{\boldsymbol{s}_0, \boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_k\} \subseteq \mathbb{Z}^d$ contains a set of collinear points. In the general case, we can assume for each i that $\boldsymbol{s}_i = (b_1 + l_i a_1, b_2 + l_i a_2, \ldots, b_d + l_i a_d)$, where $0 = l_0 < l_1 < l_2 < \cdots < l_k$, $a_i, b_i \in \mathbb{Z}$, and $\gcd(a_1, a_2, \ldots, a_d) = 1$. However since translation does not affect the polychromatic number, we will restrict our attention to the case where $a_1 > 0$ and for all i, $b_i = 0$.

Theorem 21 *Let $d \geq 2$. Let $S = \{\boldsymbol{s}_0, \boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_k\}$ be a set of $k+1$ collinear points in $\mathbb{Z}^d$ where for each i, $\boldsymbol{s}_i = (l_i a_1, l_i a_2, \ldots, l_i a_d)$, where $0 = l_0 < l_1 < l_2 < \cdots < l_k$, $a_i \in \mathbb{Z}$, $a_1 > 0$, and $\gcd(a_1, a_2, \ldots, a_d) = 1$. Let $S' = \{0, l_1, l_2, \ldots, l_k\} \subseteq \mathbb{Z}$. Then $p_{\mathbb{Z}^d}(S) = p_{\mathbb{Z}}(S')$.*

Proof: Let $S'' = \{0, l_1 a_1, l_2 a_1, \ldots, l_k a_1\} \subseteq \mathbb{Z}$. By Lemma 12, Part (i), $p_{\mathbb{Z}}(S') = p_{\mathbb{Z}}(S'')$. Since S'' can be obtained from S by a sequence of $d-1$ projections, Lemma 18 implies $p_{\mathbb{Z}^d}(S) \geq p_{\mathbb{Z}}(S'') = p_{\mathbb{Z}}(S')$. For the other direction, let $\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ and note that the function $\phi : \mathbb{Z} \to \mathbb{Z}^d$ where $\phi(n) = n\boldsymbol{a}$ is a homomorphism where $\phi(S') = S$. Thus by Lemma 9, $p_{\mathbb{Z}}(S') \geq p_{\mathbb{Z}^d}(S)$ ∎

Now we return to the subject of tilings. Lemma 22 and Theorems 23, 24, and 25 are well-known in the field of discrete geometry (see e.g. Section III of [14]) as simple examples of "splitting" groups. We restate them here using the language of polychromatic colorings.

Lemma 22 *If a set $S \subseteq G$ tiles a nontrivial subgroup H of G, then S tiles G.*

Proof: Suppose $S \subseteq G$ tiles a nontrivial subgroup H of G. Theorem 14 implies $p_H(S) = |S|$, so by Lemma 11, $p_G(S) = |S|$. By Theorem 14, S tiles G. ∎

For any $d \geq 1$, let $\boldsymbol{0}$ denote the element $(0, 0, \ldots, 0) \in \mathbb{Z}^d$ and let $\boldsymbol{e}_i$ denote the element $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^d$ with all 0's except for a 1 in the ith position. For $\boldsymbol{s} = (v_1, \ldots, v_d) \in \mathbb{Z}^d$, let $-\boldsymbol{s} = (-v_1, \ldots, -v_d)$. Define the d-semicross $SC_d = \{\boldsymbol{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ and the d-cross $C_d = \{\boldsymbol{0}, \boldsymbol{e}_1, -\boldsymbol{e}_1, \boldsymbol{e}_2, -\boldsymbol{e}_2, \ldots, \boldsymbol{e}_d, -\boldsymbol{e}_d\}$. Theorem 14 implies that any finite set $S \subseteq G$ with $p(S) = |S|$ tiles G, and we use this insight to show that these sets tile $\mathbb{Z}^d$.

Theorem 23 *For all $d \geq 1$, the d-semicross $SC_d = \{\boldsymbol{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ tiles $\mathbb{Z}^d$.*

Proof: Consider the coloring $\chi : \mathbb{Z}^d \to [d+1]$ where $\chi(v_1, \ldots, v_d) = v_1 + 2v_2 + 3v_3 + \cdots + dv_d \pmod{d+1}$. On any translate $\boldsymbol{n} + SC_d \subseteq \mathbb{Z}^d$, the colors $\chi(\boldsymbol{n} + \boldsymbol{0}), \chi(\boldsymbol{n} + \boldsymbol{e}_1), \chi(\boldsymbol{n} + \boldsymbol{e}_2), \ldots, \chi(\boldsymbol{n} + \boldsymbol{e}_d)$ are $\chi(\boldsymbol{n}), \chi(\boldsymbol{n}) + 1, \chi(\boldsymbol{n}) + 2, \ldots, \chi(\boldsymbol{n}) + d \pmod{d+1}$. They are all different, so χ is SC_d-polychromatic with $|SC_d| = d+1$ colors. By Theorem 14, SC_d tiles $\mathbb{Z}^d$. ∎

Theorem 24 *For all $d \geq 1$, the d-cross $C_d = \{\boldsymbol{0}, \boldsymbol{e}_1, -\boldsymbol{e}_1, \boldsymbol{e}_2, -\boldsymbol{e}_2, \ldots, \boldsymbol{e}_d, -\boldsymbol{e}_d\}$ tiles $\mathbb{Z}^d$.*

Proof: The $(2d+1)$-coloring $\chi : \mathbb{Z}^d \to [2d+1]$ where $\chi(v_1, \ldots, v_d) = v_1 + 2v_2 + 3v_3 + \cdots + dv_d \pmod{2d+1}$ is C_d-polychromatic: On any translate $\boldsymbol{n} + C_d \subseteq \mathbb{Z}^d$, the colors $\chi(\boldsymbol{n}+\boldsymbol{0}), \chi(\boldsymbol{n}+\boldsymbol{e}_1), \chi(\boldsymbol{n}-\boldsymbol{e}_1), \chi(\boldsymbol{n}+\boldsymbol{e}_2), \chi(\boldsymbol{n}-\boldsymbol{e}_2), \ldots, \chi(\boldsymbol{n}+\boldsymbol{e}_d), \chi(\boldsymbol{n}-\boldsymbol{e}_d)$ are $\chi(\boldsymbol{n}), \chi(\boldsymbol{n})+1, \chi(\boldsymbol{n})-1, \chi(\boldsymbol{n})+2, \chi(\boldsymbol{n})-2, \ldots, \chi(\boldsymbol{n})+d, \chi(\boldsymbol{n})-d \pmod{2d+1}$.
∎

Theorem 25 *Let $d \geq 2$. Let $S \subseteq \mathbb{Z}^d$ be a set that contains $\boldsymbol{0}$ and $j \leq d$ other elements $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_j$, where no nontrivial integer linear combination of $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_j\}$ is $\boldsymbol{0}$. Then S tiles $\mathbb{Z}^d$.*

Proof: Let $H \subseteq \mathbb{Z}^d$ be the set of all integer linear combinations of $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_j\}$. By Theorem 23, there is a set $T \subseteq \mathbb{Z}^j$ such that $\{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_j\} \oplus T = \mathbb{Z}^j$. Let $M : \mathbb{Z}^j \to \mathbb{Z}^d$ be the unique linear transformation which maps $\boldsymbol{e}_i$ to $\boldsymbol{s}_i$ for each $i \leq j$. Then $\{\mathbf{0}, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_j\}$ tiles H with complement set $\{M(t) : t \in T\}$. Since H is a subgroup of $\mathbb{Z}^d$, by Lemma 22, S tiles $\mathbb{Z}^d$. ■

We can now determine the polychromatic number of any set S of cardinality 3 or 4 in $\mathbb{Z}^d$, $d \geq 2$. Since translation does not affect polychromatic numbers, in all cases we may assume $\mathbf{0} \in S$.

Theorem 26 *Let $d \geq 2$ and suppose $S \subseteq \mathbb{Z}^d$ has cardinality 3, with $\mathbf{0} \in S$. Then $p_{\mathbb{Z}^d}(S) = 3$ if the three points are in general position or if they are collinear and there exists $S' \subseteq \mathbb{Z}$ with $p_{\mathbb{Z}}(S') = 3$ such that S' is the image of S after $d-1$ projections. Otherwise $p_{\mathbb{Z}^d}(S) = 2$.*

Proof: Theorem 25 implies that if $d \geq 2$ and $S \subseteq \mathbb{Z}^d$ consists of three points in general position, then S tiles $\mathbb{Z}^d$, and thus $p(S) = 3$. If $S \subseteq \mathbb{Z}^d$ has three collinear points, then Theorem 21 implies the problem is equivalent to finding the polychromatic number of a set of three integers, which is either 2 or 3 and can be determined using Theorem 4. ■

Theorem 27 *Let $d \geq 2$ and suppose $S \subseteq \mathbb{Z}^d$ has cardinality 4, with $\mathbf{0} \in S$. Then*

1. *If all points of S are collinear, $p_{\mathbb{Z}^d}(S)$ is 3 or 4.*
2. *If exactly three points of S are collinear, $p_{\mathbb{Z}^d}(S) = 4$.*
3. *If $d \geq 3$ and S has four points in general position, $p_{\mathbb{Z}^d}(S) = 4$.*
4. *If $d = 2$ and S has four points in general position, $p_{\mathbb{Z}^2}(S)$ is 3 or 4.*

Proof: For $d \geq 2$ and a set $S \subseteq \mathbb{Z}^d$ with $|S| = 4$, Proposition 19 implies that there is a set $S' \subseteq \mathbb{Z}$ where $|S'| = 4$ and S' can be obtained by $d-1$ projections of S. Thus Theorem 1 and Lemma 18 imply that $p(S) \geq 3$. Determining whether $p(S)$ is 3 or 4 is equivalent to determining whether S tiles $\mathbb{Z}^d$. As with the $|S| = 3$ case, we can examine cases depending on how many points of S are collinear.

If the four points of S are in general position, then if none is a nontrivial integer linear combination of the others, $p(S) = 4$ by Theorem 25. Otherwise, we can assume $S \subseteq \mathbb{Z}^2$. In this case, $p(S)$ can be 3, for example if $S = \{(0,0), (1,0), (0,1), (1,2)\} \subseteq \mathbb{Z}^2$. It can also be 4, for example if $S = \{(0,0), (1,0), (0,1), (1,1)\} \subseteq \mathbb{Z}^2$. Szegedy [15] gave an algorithm to determine if a set of cardinality 4 tiles $\mathbb{Z}^2$.

If the four points of S are all collinear, then $p(S)$ is determined by applying Theorems 21 and 4.

If exactly three of the four points are collinear, then without loss of generality assume $S = \{\mathbf{0}, (a, 0, \ldots, 0), (b, 0, \ldots 0), \boldsymbol{s}\}$, with $0 < a < b$ and $\boldsymbol{s} = (s_1, \ldots, s_d)$. Then S is in the subgroup $\{x \in \mathbb{Z}^d : x_i \in s_i\mathbb{Z} \text{ for } 2 \le i \le d\}$. By Lemmas 9 and 11 we may assume that $s_i = 1$ for some $2 \le i \le d$, and thus by a sequence of projections, S can be projected to the set $\{0, a, b, c\}$, for any $c \in \mathbb{Z}$. By Lemma 18 it suffices to show that there exists $c \in \mathbb{Z}$ such that $p_{\mathbb{Z}}(\{0, a, b, c\}) = 4$. Without loss of generality, we may assume that a and b have different parity, and in this case Theorem 4 implies that $S = \{0, a, b, a + b\}$ has polychromatic number 4. ■

The fact that $p_{\mathbb{Z}^d}(S) = 4$ if S contains exactly three collinear points implies that for any set S of three integers, there is a 4-coloring of $\mathbb{Z}$ so that every translate of S gets three different colors. Here is an explicit example of one such coloring. Without loss of generality we need only consider sets of the following form: Let $S = \{0, a, b\} \subseteq \mathbb{Z}$ where a and b are positive with a even and b odd (note that we do not specify which is larger). Define the *alternating block 4-coloring relative to S* as follows: Given any $m \in \mathbb{Z}$, let q_m and r_m be the unique integers such that $m = 2aq_m + r_m$, where $-a \le r_m < a$. Let $X(m) = 0$ if $r_m \ge 0$, $X(m) = 1$ otherwise. Let $Y(m) = 0$ if m is even, $Y(m) = 1$ otherwise. Define χ, the alternating block 4-coloring relative to S, so that $\chi(m) = (X(m), Y(m))$.

Theorem 28 *Let $S = \{0, a, b\} \subseteq \mathbb{Z}$ with $a, b > 0$, a even, and b odd. If the integers are colored with the alternating block 4-coloring relative to S then every translate of S has elements of three different colors.*

Proof: For any translate $n + S = \{n, n + a, n + b\}$ of S, $X(n) \ne X(n + a)$, while $Y(n) = Y(n + a) \ne Y(n + b)$. Thus χ has the property that any translate of S contains elements with three different colors. ■

Given a set of three integers, the alternating block 4-coloring shows that there is a 4-coloring of the integers so that every translate gets three different colors. If $S \subset \mathbb{Z}$, $|S| = 4$, is there a 5-coloring of $\mathbb{Z}$ so that every translate of S has 4 colors? More generally, we ask the following question.

Question 29 *Let $d \ge 1$. Given $k, n \in \mathbb{Z}$ with $k \le n$, let $p(n, k)$ denote the minimum r so that any $S \subseteq \mathbb{Z}$ with $|S| = n$ has an r-coloring where every translate of S gets at least k colors. What is an asymptotic upper bound on $p(n, k(n))$ for natural choices of $k(n)$?*

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