Turán numbers of vertex-disjoint cliques in r-partite graphs

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Abstract

For two graphs G and H, the *Turán number* $ex(G, H)$ is the maximum number of edges in a subgraph of G that contains no copy of H. Chen, Li, and Tu determined the Turán numbers $ex(K_{m,n}, kK_2)$ for all $k \geq 1$ [7]. In this paper we will determine the Turán numbers $ex(K_{a_1,\ldots,a_r}, kK_r)$ for all $r \geq 3$ and $k \geq 1$.

1. Introduction

All graphs considered here are finite, undirected, and simple. Throughout the paper we use the standard graph theory notation (see [1]). In particular, a graph is called a *complete r-partite graph* if its vertex set can be partitioned into r independent sets $V_1, \ldots, V_r$ such that for any $i = 1, 2, \ldots, r$ every vertex in V_i is adjacent to all other vertices in V_j, $j \neq i$. We denote a complete r-partite graph with part sizes $|V_i| = n_i$ by $K_{n_1,\ldots,n_r}$. For a graph G and a positive integer k we use kG to denote k vertex-disjoint copies of G. Given $S \subseteq V(G)$, the subgraph of G induced by S will be denoted $G[S]$ and the subgraph $G[V(G)\backslash S]$ will be denoted $G\backslash S$. For two graphs G and H, $G + H$ is the join of G and H, that is

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the graph obtained from $G \cup H$ by adding every edge containing a vertex of G and a vertex of H.

For two graphs G and H, the *Turán number*, or *extremal number*, $ex(G, H)$ is the maximum number of edges in a subgraph of a host graph G that contains no copy of H. The study of such numbers began in 1941 when Pál Turán determined the maximum number of edges in a graph on n vertices which does not contain a clique of size r, i.e. $ex(K_n, K_r)$. Since then, the most well-studied host graphs have been the complete graph K_n and the complete bipartite graph $K_{m,n}$. Recent studies of extremal numbers consider the case when the forbidden graph H is made up of several vertex-disjoint copies of some smaller graph (e.g., [2], [6], [8], [9]). In particular, Chen, Li, and Tu determined $ex(K_{m,n}, kK_2) = m(k-1)$ for $1 \leq k \leq n \leq m$ [7]. The focus of this paper is to extend their result to forbidding vertex-disjoint cliques of size r in a complete r-partite graph.

Theorem 1 (Main Theorem)**.** *For any integers* $1 \leq k \leq n_1 \leq ... \leq n_r$*,*

$$ex(K_{n_1,...,n_r}, kK_r) = \sum_{1 \leq i < j \leq r} n_i n_j - n_1 n_2 + n_2(k-1).$$

For the lower bound, clearly $((n_1 - k + 1)K_1 \cup K_{k-1,n_2}) + K_{n_3,...,n_r}$ is such a subgraph of $K_{n_1,...,n_r}$ with the required number of edges and no copy of kK_r. This gives the lower bound and the remainder of this paper will work to establish the upper bound. The upper bound is proven by considering two cases: $n_2 = n_r$ and $n_2 < n_r$. In the former case, the proof is inductive on $n_1 + k$ with Lemmas 2 and 3 as the base cases. In the latter case, the proof is inductive on the total number of vertices in the host graph.

2. Main results

The proof of Theorem 1 first considers the case when the host graph is almost balanced, that is every part size is the same except the smallest part. This proof requires a double induction and is preceded by the two necessary base cases.

Define

$$h_k(n_1, n_2, \ldots, n_r) = \sum_{1 \le i < j \le r} n_i n_j - n_1 n_2 + n_2(k-1).$$

Given two disjoint subsets of the vertex set, $A, B \subseteq V(G)$, define AB as the graph formed by the set of edges in G incident to a vertex in A and a vertex in B.

For ease of notation, given an r-partite graph G with parts $V_1, \ldots, V_r$ we let $\mathcal{R}(G,r) = \{\{v_1, \ldots, v_r\} \subseteq V(G) : v_i \in V_i \text{ for all } i \in [r]\}$, that is $\mathcal{R}(G,r)$ is the set of all r-tuples of vertices with exactly one vertex in from part. We utilize $\mathcal{R}(G,r)$ throughout to facilitate the counting of edges. For $S \in \mathcal{R}(G,r)$ define

$$w(S) = |E(G[S])|.$$

Note that for $S \in \mathcal{R}(G,r)$ an edge $v_i v_j \in V_i V_j$ is counted in $w(S)$ if and only if $v_i v_j \subseteq S$, therefore, summing over all $S \in \mathcal{R}(G,r)$,

$$\sum_{S \in \mathcal{R}(G,r)} w(S) = \sum_{1 \le i < j \le r} |E(V_i V_j)| \prod_{\ell \ne i,j} n_\ell. \tag{1}$$

Lemma 2. *For* $1 \le n_1 \le n_2$,

$$ex(K_{n_1, n_2, \ldots, n_2}, K_r) = h_1(n_1, n_2, \ldots, n_2).$$

Proof. Suppose $G \subseteq K_{n_1, n_2, \ldots, n_2}$ does not contain a copy of K_r. Then for all $S \in \mathcal{R}(G,r)$, $w(S) \le \binom{r}{2} - 1$ and hence

$$\sum_{S \in \mathcal{R}(G,r)} w(S) \le \left(\binom{r}{2} - 1\right) n_1 n_2^{r-1}. \tag{2}$$

Subtracting (2) from (1) yields,

$$
\begin{aligned}
0 &\geq \sum_{j=2}^{r} |E(V_1V_j)| n_2^{r-2} + \sum_{i,j\neq 1} |E(V_iV_j)| n_1 n_2^{r-3} - \left(\binom{r}{2} - 1\right) n_1 n_2^{r-1} \\
&= n_1 n_2^{r-3}|E(G)| + \sum_{j=2}^{r} |E(V_1V_j)| n_2^{r-3}(n_2 - n_1) - \left(\binom{r}{2} - 1\right) n_1 n_2^{r-1} \\
&\geq n_1 n_2^{r-3}|E(G)| + \left(|E(G)| - \binom{r-1}{2} n_2^2\right) n_2^{r-3}(n_2 - n_1) - \left(\binom{r}{2} - 1\right) n_1 n_2^{r-1} \\
&= n_2^{r-2}|E(G)| - \binom{r-1}{2} n_2^{r-1}(n_2 - n_1) - \left(\binom{r}{2} - 1\right) n_1 n_2^{r-1} \\
&= n_2^{r-2}|E(G)| - (r-2) n_1 n_2^{r-1} - \binom{r-1}{2} n_2^{r}.
\end{aligned}
$$

Therefore

$$
|E(G)| \leq n_1 n_2 (r-1) + \binom{r-1}{2} n_2^2 - n_1 n_2 = h_1(n_1, n_2, \ldots, n_2).
$$

□

Lemma 3. *For* $1 \leq n_1 \leq n_2$*,*

$$
ex(K_{n_1,n_2,\ldots,n_2}, n_1 K_r) = h_{n_1}(n_1, n_2, \ldots, n_2).
$$

Proof. This proof is by induction on n_1. The base case of $n_1 = 1$ is true for all positive integers n_2 by Lemma 2. Assume the statement holds for $n_1' < n_1$ where $n_1 \geq 2$. Suppose $G \subseteq K_{n_1,n_2,\ldots,n_2}$ contains more than $h_{n_1}(n_1, n_2, \ldots, n_2)$ edges and does not contain a copy of $n_1 K_r$. We have

$$
|E(G)| > h_{n_1}(n_1, n_2, \ldots, n_2) > h_1(n_1, n_2, \ldots, n_2),
$$

which implies G contains a copy of K_r. Let $S \in \mathcal{R}(G, r)$ such that $G[S] \cong K_r$. Then $|E(G \backslash S)| \leq h_{n_1-1}(n_1 - 1, n_2 - 1, \ldots, n_2 - 1)$, otherwise $G \backslash S$ contains a copy of $(n_1 - 1)K_r$, and this together with S is a copy of $n_1 K_r$ in G. Therefore,

$$
\begin{aligned}
|E(G)| - |E(G \backslash S)| &> h_{n_1}(n_1, n_2, \ldots, n_2) - h_{n_1-1}(n_1 - 1, n_2 - 1, \ldots, n_2 - 1) \\
&= (r-1)(n_1 + (r-2)n_2) + (r-1)n_2 - \binom{r}{2} - 1.
\end{aligned}
$$

Since the number of edges with a vertex in S is

$$(r-1)(n_1+(r-2)n_2)+(r-1)n_2-\binom{r}{2},$$

this implies all edges in the host graph containing a vertex in S are present in G. Note that this holds for every S such that $G[S]=K_r$ in G.

Let $u_i \in V_i$ and $u_j \in V_j$ with $i \neq j$, if either u_i or u_j is in S, then $u_iu_j \in E(G)$. Otherwise, for $v_i \in S \cap V_i$, let $S'=(S\backslash\{v_i\})\cup\{u_i\}$. S' induces a copy of K_r in G and therefore $u_i, u_j \in E(G)$. Hence $G \cong K_{n_1,n_2,\dots,n_2}$ and thus contains n_1K_r which is a contradiction. □

We now prove Theorem 1. The proof considers two cases: $n_2 = n_r$ and $n_2 < n_r$. The first case is proven using double induction and relies on Lemmas 2 and 3 as the base cases. The second case is proven by induction on the total number of vertices in the host graph.

Proof. Let $1 \leq k \leq n_1 \leq \cdots \leq n_r$.

Case 1. Assume $n_2 = n_r$, we proceed by induction on $n_1 + k$. The base case of $k = 1$ is true for all positive integers n_1 by Lemma 2 and the base case of $n_1 = k$ is true for all positive integers $k \leq n_1$ by Lemma 3. Assume the statement is true for parameters n_1', k' such that $n_1' + k' < n_1 + k$ where $n_1 > k \geq 2$, and that $G \subseteq K_{n_1,n_2,\dots,n_2}$ does not contain a copy of kK_r.

We first obtain a lower bound on the number of (not necessarily disjoint) copies of K_r in G. Suppose there are exactly q such copies of K_r in G, then

$$\sum_{S\in\mathcal{R}(G,r)} w(S) \leq q\binom{r}{2} + (n_1n_2^{r-1}-q)\left(\binom{r}{2}-1\right).$$

Recall that

$$\sum_{S\in\mathcal{R}(G,r)} w(S) = \sum_{j=2}^{r}|E(V_1V_j)|n_2^{r-2} + \sum_{i,j\neq 1}|E(V_iV_j)|n_1n_2^{r-3},$$

this gives,

$$q \geq \sum_{j=2}^{r}|E(V_1V_j)|n_2^{r-2} + \sum_{i,j\neq 1}|E(V_iV_j)|n_1n_2^{r-3} - n_1n_2^{r-1}\left(\binom{r}{2}-1\right). \quad (3)$$

We will use (3) to get an upper bound on $|E(G)|$ by counting $\sum_{S\in\mathcal{R}(G,r)}|E(G\backslash S)|$. An edge $v_iv_j \in V_iV_j$ is counted in $|E(G\setminus S)|$ if and only if $v_i \notin S$ and $v_j \notin S$, hence

$$\sum_{S\in\mathcal{R}(G,r)}|E(G\backslash S)| = \sum_{i=2}^{r}|E(V_1V_j)|(n_1-1)(n_2-1)n_2^{r-2}+\sum_{i,j\neq 1}|E(V_iV_j)|(n_2-1)^2n_1n_2^{r-3}. \quad (4)$$

Using (3) and (4),

$$\begin{aligned}\sum_{S\in\mathcal{R}(G,r)}|E(G\backslash S)| + \Big(q+n_1n_2^{r-1}\left(\tbinom{r}{2}-1\right)\Big)(n_2-1) &\geq \sum_{j=2}^{r}|E(V_1V_j)|n_1(n_2-1)n_2^{r-2}\\ &+\sum_{i,j\neq 1}|E(V_iV_j)|(n_2-1)n_1n_2^{r-1}\\ &=|E(G)|(n_2-1)n_2^{r-2}n_1. \qquad (5)\end{aligned}$$

Now for $S\in\mathcal{R}(G,r)$, suppose $G[S]$ is a copy of K_r. Then $|E(G\backslash S)| \leq h_{k-1}(n_1-1,n_2-1,\ldots,n_2-1)$ else by induction $G\backslash S$ contains a copy of $(k-1)K_r$ and so this together with S yields a copy of kK_r in G. If $G[S]$ is not complete, then since $G\backslash S$ does not contain a copy of kK_r induction gives $|E(G\backslash S)| \leq h_k(n_1-1,n_2-1,\ldots,n_2-1)$. Hence

$$\begin{aligned}\sum_{S\in\mathcal{R}(G,r)}|E(G\setminus S)| &\leq q\big(h_{k-1}(n_1-1,n_2-1,\ldots,n_2-1)\big)\\ &+(n_1n_2^{r-1}-q)\big(h_k(n_1-1,n_2-1,\ldots,n_2-1)\big)\\ &=q(1-n_2)+n_1n_2^{r-1}(h_k(n_1-1,n_2-1,\ldots,n_2-1))\end{aligned}$$

and thus, using (5) we have

$$|E(G)|(n_2-1)n_2^{r-2}n_1 \leq n_1n_2^{r-1}\left(h_k(n_1-1,n_2-1,\ldots,n_2-1)+\left(\tbinom{r}{2}-1\right)(n_2-1)\right).$$

Therefore

$$\begin{aligned}|E(G)| &\leq \frac{n_2}{n_2-1}\left(h_k(n_1-1,n_2-1,\ldots,n_2-1)+\left(\binom{r}{2}-1\right)(n_2-1)\right)\\ &=h_k(n_1,n_2,n_2,\ldots,n_2).\end{aligned}$$

Case 2. Assume $n_2 < n_r$, we proceed by induction on the number of total vertices. The base case of $n_1 = n_r$ is true for all positive integers k by Case 1. Assume the statement holds for all parameters $n_1',\ldots,n_r'$ such that $\sum_{i=1}^{r} n_i' <$

$\sum_{i=1}^{r} n_i$. Suppose $G \subseteq K_{n_1,\ldots,n_r}$ does not contain a copy of kK_r. Let $v_r \in V_r$, the graph $G\backslash\{v_r\}$ does not contain a copy of kK_r, has fewer vertices than G, and $n_2 \leq n_r - 1$. Therefore

$$\begin{aligned}
|E(G)| &= |E(G \setminus \{v_r\})| + d(v_r) \\
&\leq ex(K_{n_1,\ldots,n_r-1}, kK_r) + d(v_r) \\
&= h_k(n_1, \ldots, n_r - 1) + d(v_r) \\
&= \sum_{1 \leq i < j \leq r} n_i n_j - \sum_{i=1}^{r-1} n_i - n_1 n_2 + n_2(k-1) + d(v_r) \\
&\leq \sum_{1 \leq i < j \leq r} n_i n_j - n_1 n_2 + n_2(k-1) \\
&= h_k(n_1, n_2, \ldots, n_r).
\end{aligned}$$

□

3. Concluding remarks

The main theorem relies on the fact that both K_r and $K_{n_1,n_2,\ldots n_r}$ are r-partite. Certainly the host graph must be ℓ-partite for $\ell > r$ to have K_r as a subgraph. An interesting generalization would be to calculate $ex(K_{n_1,n_2,\ldots,n_\ell}, kK_r)$ for $r < \ell$. In [3], De Silva, Heysse, and Young proved that

$$ex(K_{n_1,n_2,\ldots,n_\ell}, kK_2) = (k-1)\left(\sum_{i=2}^{\ell} n_i\right),$$

however the Turán number is open for $r \geq 3$. The graph

$$((n_1 + n_2 - k + 1)K_1 \cup K_{k-1,n_3}) + n_4 K_1$$

does not contain kK_3, hence

$$ex(K_{n_1,n_2,n_3,n_4}, kK_3) \geq (n_1 + n_2 + n_3)n_4 + (k-1)n_3.$$

This construction can be easily generalized to r-partite graphs, but it is not clear that this is an extremal construction.

We also note that many of the results cited in this paper were originally considered in conjunction with the *rainbow number* $rb(G, H)$. For a graph G and subgraph H, $rb(G, H)$ is the minimum number of colors required to ensure that every edge coloring of G with $rb(G, H)$ colors has a rainbow copy of H (where a subgraph is *rainbow* if it has no two edges with the same color). Often the rainbow number is proven via the analogous Turán number, and it would be interesting to see this work extended to the rainbow number $rb(K_{n_1,\ldots n_r}, kK_r)$.

4. References

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