Polychromatic colorings of complete graphs with respect to 1-, 2-factors and Hamiltonian cycles

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December 10, 2016

Abstract

If G is a graph and $\mathcal{H}$ is a set of subgraphs of G, then an edge-coloring of G is called $\mathcal{H}$-polychromatic if every graph from $\mathcal{H}$ gets all colors present in G on its edges. The $\mathcal{H}$-polychromatic number of G, denoted $\text{poly}_{\mathcal{H}}(G)$, is the largest number of colors in an $\mathcal{H}$-polychromatic coloring. In this paper, $\text{poly}_{\mathcal{H}}(G)$ is determined exactly when G is a complete graph and $\mathcal{H}$ is the family of all 1-factors. In addition $\text{poly}_{\mathcal{H}}(G)$ is found up to an additive constant term when G is a complete graph and $\mathcal{H}$ is the family of all 2-factors, or the family of all Hamiltonian cycles.

1 Introduction

If G is a graph and $\mathcal{H}$ is a set of subgraphs of G, we say that an edge-coloring of G is *$\mathcal{H}$-polychromatic* if every graph from $\mathcal{H}$ gets all colors present in G on its edges. The *$\mathcal{H}$-polychromatic number of G*, denoted $\text{poly}_{\mathcal{H}}(G)$, is the largest number of colors in an $\mathcal{H}$-polychromatic coloring. If an $\mathcal{H}$-polychromatic coloring of G uses $\text{poly}_{\mathcal{H}}(G)$ colors, it is called an *optimal* $\mathcal{H}$-polychromatic coloring of G.

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1.1 Background

Let Q_n denote the hypercube of dimension n. Let $G = Q_n$ and $\mathcal{H}$ be the family of all subgraphs of G isomorphic to Q_d. If d is fixed and n is large, then Alon, Krech, and Szabó [3] showed that $\lfloor \frac{(d+1)^2}{4} \rfloor \leq \text{poly}_{\mathcal{H}}(Q_n) \leq \binom{d+1}{2}$. Offner [11] proved that the lower bound is tight for all sufficiently large values of n. Bialostocki [4] treated the special case when $d = 2$ and $n \geq 2$. Goldwasser *et al.* [9] considered the case where $\mathcal{H}$ is the family of all subgraphs of Q_n isomorphic to a Q_d minus an edge or a Q_d minus a vertex.

If T is a tree and $\mathcal{H}$ is the set of all paths of length at least r, then $\text{poly}_{\mathcal{H}}(T) = \lceil r/2 \rceil$, as was shown by Bollobás *et al.* [5]. When $G = K_n$ and $\mathcal{H}$ is the set of all r-vertex cliques, $\text{poly}_{\mathcal{H}}(G)$ was considered by Erdős and Gyárfás [6, 10] with the respective colorings called balanced. When G is an arbitrary multigraph of minimum degree d, and $\mathcal{H}$ is the set of all stars with center v and leaves $N(v)$, $v \in V(G)$, then it was shown by Alon *et al.* [2], that $\text{poly}_{\mathcal{H}}(G) \geq \lfloor (3d+1)/4 \rfloor$. Goddard and Henning [7] considered vertex-colorings of graphs such that each open neighborhood contains a vertex of every color used in G.

Polychromatic colorings were also investigated for vertex-colored hypergraphs. These colorings are essential tools in studying covering problems which are of fundamental importance in general graph and hypergraph settings, especially in geometric hypergraphs, and they exhibit connections to VC-dimension, see [1, 2, 5, 12].

1.2 Main Results

In this paper, we consider the case where G is a complete graph and $\mathcal{H}$ is a family of spanning subgraphs. Let $\mathrm{F}_1 = \mathrm{F}_1(n)$ be the family of all 1-factors of K_n, $\mathrm{F}_2 = \mathrm{F}_2(n)$ be the family of all 2-factors of K_n and $\mathrm{HC} = \mathrm{HC}(n)$ be the family of all Hamiltonian cycles of K_n. Our main results are as follows:

Theorem 1. *If n is an even positive integer, then* $\text{poly}_{\mathrm{F}_1}(K_n) = \lfloor \log_2 n \rfloor$.

Theorem 2. *There exists a constant c such that* $\lfloor \log_2 2(n+1) \rfloor \leq \text{poly}_{\mathrm{F}_2}(K_n) \leq \text{poly}_{\mathrm{HC}}(K_n) \leq \lfloor \log_2 n \rfloor + c$. *Moreover,* $\left\lfloor \log_2 \frac{8(n-1)}{3} \right\rfloor \leq \text{poly}_{\mathrm{HC}}(K_n)$.

It is claimed in a follow-up paper [8], that in fact $\text{poly}_{\mathrm{F}_2}(K_n) = \lfloor \log_2 2(n+1) \rfloor$ and $\text{poly}_{\mathrm{HC}}(K_n) = \left\lfloor \log_2 \frac{8(n-1)}{3} \right\rfloor$ for $n \geq 3$. However, the arguments there include more case analysis and greater detail than what is required for the small additive constant given in Theorem 2.

The paper is structured as follows. We start with basic definitions in Section 2. In Section 3, we give constructions of polychromatic colorings, which provide the lower bounds for Theorems 1 and 2. In Section 4, we prove Theorem 1. Section 5 contains the proof of Theorem 2.

2 Definitions

Let the vertices of K_n be denoted by $v_1, v_2, \ldots, v_n$. An edge-coloring φ is *ordered at* v_i for $i \in [n]$ if there exists a color a, called the *main color at* v_i, such that $\varphi(v_i v_j) = a$ for all $j \in \{i+1, \ldots, n\}$. Notice that v_{n-1} and v_n are ordered with respect to any coloring. We define the main color of v_n to be the same as for v_{n-1}. A vertex v_i is *unitary* if there are colors $a \neq b$ such that v_i is incident with $n-2$ edges colored a and one edge $v_i v_j$ colored b, where v_j is unitary with $n-2$ incident edges colored b. For v_i unitary, we also call a the *main color*.

An edge-coloring is *ordered* if all vertices are ordered with respect to some ordering of vertices. See Figure 1 for an example of an ordered coloring. We call an edge-coloring *combed* if each vertex is either ordered or unitary. It is not difficult to show that if there is at least one unitary vertex in a combed coloring then either the first three vertices (and no others) are unitary with different main colors, as in Figure 2, or the first four vertices (and no others) are unitary with two of them with one main color, and two with another.

Let φ be an ordered or combed coloring. The *inherited coloring* is the vertex-coloring φ' obtained by coloring each vertex with its main color. Its *inherited color class* M_i *of color* i is the set of all vertices v with $\varphi'(v) = i$. Let $M_t(j) = M_t \cap \{v_1, v_2, \ldots, v_j\}$. In this paper, we shall always think of the ordered vertices as arranged on a horizontal line with v_i to the left of v_j if $i < j$. We say that an edge $v_i v_j$, $i < j$, goes from v_i to the right and from v_j to the left. If φ is an edge-coloring of a graph G, the *maximum monochromatic degree* of G is the largest integer d such that some vertex of G is incident to d edges of the same color. We say such a vertex is a *max-vertex*. If X is a subset of $V(K_n)$, we say that the edge-coloring φ of K_n is

- X*-constant* if for any $v \in X$, $\varphi(vu) = \varphi(vw)$ for all $u, w \in V \setminus X$,
- X*-ordered* if there is an ordering of the vertices such that $X = \{v_1, \ldots, v_m\}$ for some integer m and φ is ordered on vertices in X.

Notice if a coloring φ is X-ordered, it is also X-constant.

3 Constructions of Polychromatic Colorings

We construct three edge-colorings of K_n, and show that they are polychromatic for F_1, F_2, and HC, respectively.

3.1 F_1-polychromatic Coloring φ_{F_1}

Let $n \geq 2$ be even, and let k be the largest positive integer such that $2^k \leq n$, i.e., $k = \lfloor \log_2 n \rfloor$. Let φ' be a vertex-coloring of K_n with vertex set $\{v_1, \ldots, v_n\}$ and colors $1, \ldots, k$, where for

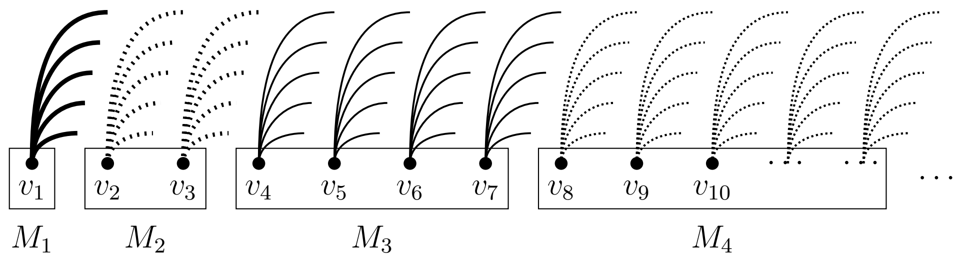

Figure 1: F_1-polychromatic coloring φ_{F_1}.

each $i \in [k]$, M_i is the color class of color i. Moreover, for any $1 \leq i < j \leq k$, every vertex in M_i precedes every vertex in M_j, and $|M_t| = 2^{t-1}$ for $t = 1, \ldots, k-1$. Hence the color classes $1, 2, \ldots, k$ have sizes $1, 2, 4, \ldots, 2^{k-2}, n-2^{k-1}+1$, respectively. Let φ_{F_1} be the ordered coloring for which φ' is the inherited coloring.

Consider an arbitrary 1-factor F of K_n and $t \in [k]$. Consider the set F_t of all edges of F with at least one endpoint in M_t. Since $|M_1| + \cdots + |M_i| = 2^i - 1$ and $|M_k| = n - |M_1| - \cdots - |M_{k-1}| \geq 2^k - 2^{k-1} + 1 = 2^{k-1} + 1$, we have $\sum_{i<t} |M_i| < |M_t|$ for any $t \in [k]$. Thus at least one edge of F_t joins a vertex from M_t to a vertex to the right, so this edge is of color t. Therefore F has edges of each color. Hence φ_{F_1} is F_1-polychromatic and it uses $\lfloor \log_2 n \rfloor$ colors.

3.2 F_2-polychromatic Coloring φ_{F_2}

Let k be the largest positive integer such that $n \geq 2^{k-1}-1$, i.e., $k = 1+\lfloor \log_2(n+1) \rfloor$. Let φ' be a vertex-coloring of K_n with vertex set $\{v_1, \ldots, v_n\}$ and colors $1, \ldots, k$, where for each $i \in [k]$, M_i is the color class of color i. Moreover, for any $1 \leq i < j \leq k$, every vertex in M_i precedes every vertex in M_j, and $|M_t| = 2^{t-2}$ for $t = 4, \ldots, k-1$, and $|M_1| = |M_2| = |M_3| = 1$. Hence the color classes $1, 2, \ldots, k-1, k$ have sizes $1, 1, 1, 4, 8, \ldots, 2^{k-3}, n - 2^{k-2} + 1$, respectively. Let φ_{F_2} be obtained by taking the ordered coloring for which φ' is the inherited coloring and then recoloring the edge v_1v_3 from color 1 to color 3. See Figure 2.

Observe that the inherited color classes M_1, M_2, and M_3 contain unitary vertices. Moreover, $|M_1| + \cdots + |M_t| = 2^{t-1} - 1$ for $3 \leq t \leq k-1$, and $|M_k| = n - |M_1| - \cdots - |M_{k-1}| \geq 2^{k-1} - 1 - 2^{k-2} + 1 = 2^{k-2}$. So, $|M_t| > \sum_{i<t} |M_i|$ for any $t \geq 4$. Consider an arbitrary 2-factor F of K_n and $t \in [k]$. For $i \leq 3$, v_i is a unitary vertex with main color i, so F must have edges of colors 1, 2, and 3. For a color $t \geq 4$, consider the set F_t of edges of F with endpoints in M_t. Then F_t has an edge of color t unless F_t forms a bipartite graph G_t with one part M_t and another $M'_t = \bigcup_{i=1}^{t-1} M_i$. The degree of each vertex of G_t from M_t is two, and the degree of each vertex of G_t from M'_t is at most two. Thus $|M'_t| \geq |M_t|$, a contradiction. Thus, F_t, and therefore F, has at least one edge of color t. So, φ_{F_2} is F_2-polychromatic and

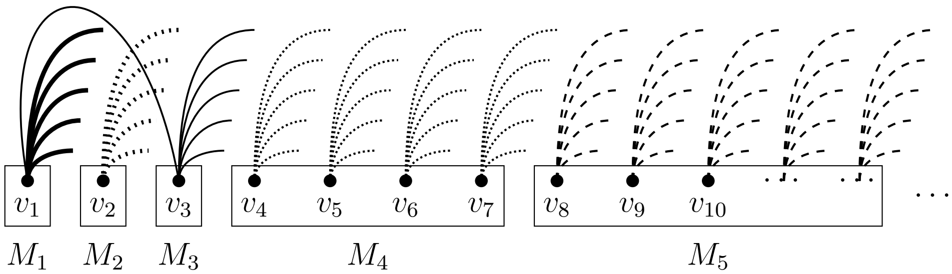

Figure 2: F_2-polychromatic coloring φ_{F_2}.

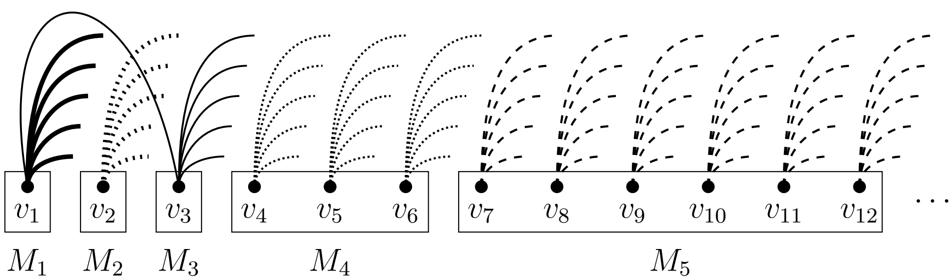

Figure 3: HC-polychromatic coloring φ_{HC}.

it uses $k = \lfloor \log_2 2(n+1) \rfloor$ colors.

3.3 HC-polychromatic Coloring φ_{HC}

Let k be the largest positive integer such that $n \geq 3 \cdot 2^{k-3} + 1$, i.e., $k = 3 + \lfloor \log_2 (n-1)/3 \rfloor$. Let φ' be a vertex-coloring of K_n with vertex set $\{v_1, \dots, v_n\}$ and colors $1, \dots, k$, where for each $i \in [k]$, M_i is the color class of color i. Moreover, for any $1 \leq i < j \leq k$, every vertex in M_i precedes every vertex in M_j, and $|M_t| = 3 \cdot 2^{t-4}$ for $t = 4, \dots, k-1$, and $|M_1| = |M_2| = |M_3| = 1$. Hence the color classes $1, 2, \dots, k-1, k$ have sizes $1, 1, 1, 3, 6, 12, \dots, 3 \cdot 2^{k-5}, n - 3 \cdot 2^{k-4}$, respectively. Let φ_{HC} be obtained by taking the ordered coloring for which φ' is the inherited coloring and then recoloring the edge $v_1 v_3$ from color 1 to color 3. See Figure 3.

We have that $|M_1| + \cdots + |M_t| = 3 \cdot 2^{t-3}$ for $3 \leq t \leq k-1$. Moreover, $|M_k| = n - |M_1| - \cdots - |M_{k-1}| \geq 3 \cdot 2^{k-3} + 1 - 3 \cdot 2^{k-4} = 3 \cdot 2^{k-4} + 1$. Thus $|M_k| > \sum_{i<k} |M_i|$ and $|M_t| \geq \sum_{i<t} |M_i|$ for all $t \geq 4$. Consider an arbitrary Hamiltonian cycle H of K_n. For $i \leq 3$, v_i is a unitary vertex with main color i, so H must have edges of colors 1, 2, and 3. For each color $t \geq 4$, let H_t be the set of edges of H with at least one endpoint in M_t.

Then H_t has an edge of color t unless H_t forms a bipartite graph G_t with one part M_t and another $M'_t = \bigcup_{i=1}^{t-1} M_i$. The degree of each vertex of G_t from M_t is two, and the degree of each vertex of G_t from M'_t is at most two. If $4 \leq t < k$, $|M_t| = |M'_t|$, the degree of each vertex of G_t from M'_t is also two. Hence G_t is a union of cycles, so it could not be a proper subgraph of a Hamiltonian cycle. If $t = k$, $|M_k| > |M'_k|$, so a bipartite graph G_t could not exist. Thus H has an edge of color t for each $t = 1, \ldots, k$, φ_{HC} is HC-polychromatic, and it uses $\left\lfloor \log_2 \frac{8(n-1)}{3} \right\rfloor$ colors.

4 Proof of Theorem 1

We prove Theorem 1 by first showing the existence of an optimal edge-coloring that is ordered. Then we use Lemma 3 below which states that, for every inherited color class M_t, there exists j such that a majority of $v_1, \ldots, v_j$ is in M_t. This leads to a counting argument that gives the upper bound in Theorem 1. For the lower bound we use the coloring φ_{F_1}.

Lemma 3. *Let $\varphi : E(K_n) \to [k]$, where n is even, be an ordered coloring with inherited color classes $M_1, \ldots, M_k$. If the coloring φ is F_1-polychromatic, then $\forall t \in [k]\ \exists j \in [n-1]$ such that $|M_t(j)| > j/2$.*

Proof. Assume there exists t such that for each $j \in [n-1]$, $|M_t(j)| \leq j/2$. Let $x_1, \ldots, x_m$ be the vertices of M_t in order and let $y_1, \ldots, y_{n-m}$ be the other vertices of K_n in order. Let H consist of the edges $y_1x_1, y_2x_2, \ldots, y_mx_m$ and a perfect matching on $\{y_{m+1}, \ldots, y_{n-m}\}$ (if this set is non-empty). Since $|M_t(j)| \leq j/2$ for all j, the number of y's that must precede x_i is at least i for each $i = 1, \ldots, m$. Hence y_i is to the left of x_i for each $i = 1, \ldots, m$. Therefore all edges in H incident with vertices in M_t go to the left and do not have color t. The edges of H that are not incident with vertices in M_t are also not of color t. Hence φ is not F_1-polychromatic, a contradiction. □

Proof of Theorem 1. Let $k = \mathrm{poly}_{\mathrm{F}_1}(K_n)$ be the polychromatic number for 1-factors in $K_n = (V, E)$. Among all F_1-polychromatic colorings of K_n with k colors we choose ones that are X-ordered for a subset X (possibly empty) of the largest size, and, of these, choose a coloring φ whose restriction to $V \setminus X$ has the largest maximum monochromatic degree. Suppose for contradiction that $V \neq X$.

Let $Z = V \setminus X$ and G be the subgraph of K_n induced by Z. Let v be a vertex of maximum monochromatic degree, d, in φ restricted to G, and let 1 be a color for which there are d edges incident with v in G with color 1. By the maximality of $|X|$, there is a vertex u in Z such that $\varphi(uv) \neq 1$. Assume $\varphi(uv) = 2$. If every 1-factor containing uv had another edge of color 2, then the color of uv could be changed to 1, resulting in an F_1-polychromatic coloring where v has a larger maximum monochromatic degree in G, a contradiction. Hence, there is a 1-factor F in which uv is the only edge with color 2 in φ.

Let $\varphi(vy_i) = 1$, $y_i \in Z$, $i = 1, \ldots, d$. For each $i \in [d]$, let $y_i w_i$ be the edge of F containing y_i (perhaps $w_i = y_j$ for some $j \neq i$); see Figure 4. We can get a different 1-factor F_i by replacing the edges uv and $y_i w_i$ in F with edges vy_i and uw_i. Since F_i must have an edge of color 2 and $\varphi(vy_i) = 1$, we must have $\varphi(uw_i) = 2$ for each $i \in [d]$.

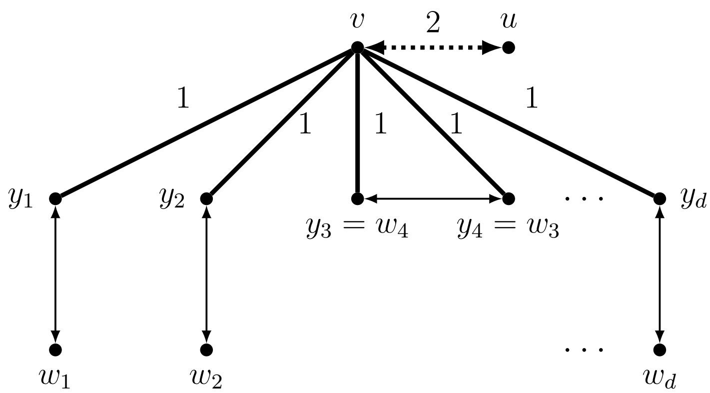

Figure 4: Maximum polychromatic degree in an F_1-polychromatic coloring.

If $w_i \in X$ for some i then, since φ is X-constant, $\varphi(w_i y_i) = \varphi(w_i u) = 2$, so $y_i w_i$ and uv are two edges of color 2 in F, a contradiction. So, $w_i \in Z$ for all $i \in [d]$. Thus $\varphi(uv) = \varphi(uw_1) = \cdots = \varphi(uw_d) = 2$, and the monochromatic degree of u in G is at least $d+1$, larger than that of v, a contradiction.

We conclude that $X = V$. Hence φ is an ordered F_1-polychromatic coloring of K_n. By Lemma 3, for every $t \in [k]$ there exists j_t such that $|M_t(j_t)| > \frac{j_t}{2}$. By permuting the colors, we can assume $j_{t_1} < j_{t_2}$ whenever $t_1 < t_2$. This gives us an ordering of inherited color classes $M_1, M_2, \ldots, M_k$. Since $|M_1| \geq 1$ and $|M_t(j_t)| > \frac{j_t}{2}$, we can use induction to show $|M_t| \geq |M_t(j_t)| \geq 2^{t-1}$ as follows

$$|M_t| \geq |M_t(j_t)| > \sum_{1 \leq i < t} |M_i(j_t)| \geq \sum_{1 \leq i < t} |M_i(j_i)| \geq \sum_{1 \leq i < t} 2^{i-1} = 2^{t-1} - 1.$$

The sum of the sizes of all inherited color classes is n, and we get

$$n = \sum_{t=1}^{k} |M_t| \geq \sum_{t=1}^{k} 2^{t-1} = 2^k - 1.$$

Since n is even, $n \geq 2^k$ and $\mathrm{poly}_{F_1}(K_n) = k \leq \lfloor \log_2 n \rfloor$.

The fact that $\mathrm{poly}_{F_1}(K_n) \geq \lfloor \log_2 n \rfloor$ follows from the coloring φ_{F_1}. This finishes the proof of Theorem 1. □

5 Proof of Theorem 2

Recall that we call an edge-coloring φ *combed* if all vertices are either ordered or unitary.

We prove Theorem 2 by first showing the existence of an optimal edge-coloring that is combed. Then we use Lemma 4 below which states that, for every inherited color class M_t, either there exists j such that at least half of $v_1, \ldots, v_j$ is in M_t or M_t contains a unitary vertex. This leads to a counting argument that finishes the proof of Theorem 2.

Lemma 4. *Let $\varphi : E(K_n) \to [k]$ be a combed coloring with inherited color classes $M_1, \ldots, M_k$. If the coloring φ is F_2-polychromatic, or HC-polychromatic, then $\forall t \in [k]$ $\exists j \in [n-1]$ such that $|M_t(j)| \geq \frac{j}{2}$ or M_t contains a unitary vertex.*

Proof. Let $\mathcal{H} \in \{\mathrm{F}_2, \mathrm{HC}\}$. Let φ be a combed $\mathcal{H}$–polychromatic coloring with inherited color classes $M_1, \ldots, M_k$. It is sufficient to consider an arbitrary color $t \in [k]$ and show that the condition on M_t is satisfied.

Let $x_1, \ldots, x_m$ be the vertices of M_t in order and let $y_1, \ldots, y_{n-m}$ be the other vertices of K_n in order. Suppose for contradiction that there exists t such that $|M_t(j)| < \frac{j}{2}$ for all $j \in [n-1]$ and M_t does not contain a unitary vertex. Thus φ is ordered at each $x_i \in M_t$ and so y_{i+1} is to the left of x_i for each $i \in [m]$. Consider a Hamiltonian cycle $H = y_1 x_1 y_2 x_2 \cdots y_m x_m y_{m+1} \cdots y_{n-m} y_1$.

Since $|M_t(j)| < j/2$ for all j, the number of y's that must precede x_i is at least $i+1$ for each $i = 1, \ldots, m$. Hence y_i and y_{i+1} are to the left of x_i for each $i = 1, \ldots, m$. Therefore each edge in H incident with a vertex x_i in M_t goes to the left from the perspective of x_i.

Let yx be an edge of H, where $x \in M_t$. Since $y \notin M_t$, the majority color r of y is not t. Since φ is combed, either $\varphi(yx) = r$ or $\varphi(yx) \neq r$ and both y and x are unitary vertices. Recall M_t does not contain any unitary vertices. Hence no edge in H is colored by t. This contradicts the fact that φ is $\mathcal{H}$-polychromatic. □

We say that a Hamiltonian cycle H' is obtained from a Hamiltonian cycle H by a *twist* of disjoint edges e_1 and e_2 of H if $E(H) \setminus \{e_1, e_2\} \subseteq E(H')$, i.e. we remove e_1, e_2 from H and introduce two new edges to make the resulting graph a Hamiltonian cycle. Note that the choice of these two edges to add is unique. The other choice of two edges to add does not preserve connectivity. Without the connectivity requirement, the operation is known as a *2-switch*.

Notice that any 2-switch could be applied to a 2-factor and the result will be again a 2-factor. Here, it might be possible to add the two new edges in two different ways.

For $\mathcal{H} \in \{\mathrm{HC}, \mathrm{F}_2\}$, Lemma 5 can be used to show that there exists an optimal $\mathcal{H}$-polychromatic that is combed.

Lemma 5. *Suppose $n \geq 3$ and $X \subset V(K_n)$. Let $\mathcal{H} \in \{\mathrm{HC}, \mathrm{F}_2\}$ and φ_1 be an optimal $\mathcal{H}$-polychromatic coloring of K_n that is X-constant. Then there exists an optimal $\mathcal{H}$ polychromatic coloring φ of K_n that agrees with φ_1 on all edges with at least one endpoint in X such that*

(A) there exists a vertex $v \in V(K_n) \setminus X$ such that φ is $(X \cup \{v\})$-constant; or

(B) $X = \emptyset$ and there exist vertices x, y, z, such that these vertices are unitary under φ of distinct main colors. This implies φ is $\{x, y, z\}$-constant and xyz is a rainbow triangle.

Proof. Let $\mathcal{H} \in \{\mathrm{F}_2, \mathrm{HC}\}$. Let $Z = V(K_n) \setminus X$ and G be the subgraph of K_n induced by Z. Let $|Z| = m$. If $m \leq 2$ then $X \neq \emptyset$ and (A) is trivially satisfied. Hence $m \geq 3$. If $X = \emptyset$ and there exists an optimal $\mathcal{H}$-polychromatic coloring φ with three unitary vertices x, y, and z of distinct main colors, then (B) is satisfied. Hence we assume there is no such edge-coloring φ.

Choose φ to be an optimal $\mathcal{H}$-polychromatic coloring such that it agrees with φ_1 on edges with at least one endpoint in X and subject to this, it maximizes the maximum monochromatic degree of G. Define d to be the maximum monochromatic degree of vertices in G with φ.

First suppose $d = m - 1$. Let v be a vertex of maximum monochromatic degree d in G. Then φ is $(X \cup \{v\})$-constant and we have (A). Hence we assume $d \leq m - 2$.

Let $\ell = m - 1 - d$ and let φ use colors $1, \ldots, k$. If color a appears d times in G at a vertex $v \in Z$, we say v is an *a-max-vertex*. If the ℓ edges incident with v in G which do not have color a all have color b, we call v an *(a, b)-max-vertex* with *minority color* b.

Claim 1. If $a, b \in [k]$ are two distinct colors, $v \in Z$ is an a-max-vertex and $\varphi(vu) = b$ for some other vertex $u \in Z$, then all of the following hold:

(1) u is a b-max-vertex,

(2) v is an (a, b)-max-vertex,

(3) either $X = \emptyset$ or $\mathcal{H} = \mathrm{F}_2$, and

(4) at least half of the edges between X and Z have color b.

Proof. For ease of notation, we assume that $a = 1$ and $b = 2$. Let v be a 1-max-vertex. Let $u \in Z$ be a vertex such that $\varphi(vu) = 2$. If every $H \in \mathcal{H}$ containing uv contains another edge of color 2, we could change the color of uv to color 1, giving an $\mathcal{H}$-polychromatic coloring where v has monochromatic degree $d + 1$, a contradiction. Hence, there must be $H \in \mathcal{H}$ where uv is the only edge of color 2.

Cyclically orient the edges of each cycle in H such that uv is an arc, and denote the resulting directed graph $\overrightarrow{H}$. Let $\varphi(vy_j) = 1$, for $y_j \in Z$, $j = 1, 2, \ldots, d$. For each $j \in [d]$, let w_j be the predecessor of y_j in $\overrightarrow{H}$, so $\overrightarrow{w_j y_j} \in \overrightarrow{H}$ for each j. We assume $w_j \neq v$ for $j = 2, 3, \ldots, d$, but perhaps $w_1 = v$ and perhaps $w_j = y_i$ for some $i \neq j$. See Figure 5.

Now we shall prove (1). If $w_j \neq v$, twist the edges uv and $w_j y_j$ of H to get a new $H_j \in \mathcal{H}$ containing vy_j and uw_j. Since H_j must have an edge of color 2 and $\varphi(vy_j) = 1$,

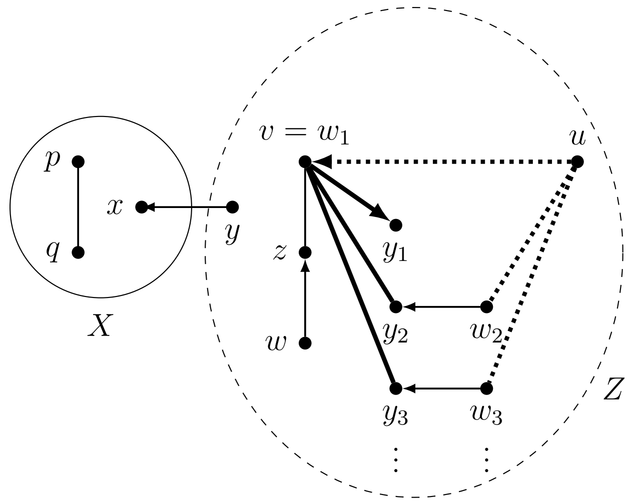

Figure 5: Situation in Claim 1.

we must have $\varphi(uw_j) = 2$. Hence, $\varphi(uv) = \varphi(uw_2) = \varphi(uw_3) = \cdots = \varphi(uw_d) = 2$. Note that $w_j \in Z$ for each $j \in [d]$. This is because if $w_j \in X$, then, since φ is X-constant and $y_j \in Z$, $\varphi(w_j y_j) = \varphi(w_j u) = 2$, so $w_j y_j$ is another edge in H with color 2, a contradiction. That gives us d edges of color 2 at u in G. Note that if $w_1 \neq v$, then uw_1 is another edge of color 2 incident to u, so $w_1 = v$ and $\overrightarrow{vy_1}$ is an arc of $\overrightarrow{H}$. Therefore, u is a 2-max-vertex. This proves (1).

Next, we prove (2), i.e., that v is a $(1,2)$-max-vertex. Let $z \in Z$ be a vertex distinct from u such that $\varphi(vz) \neq 1$. Let w be the vertex such that $\overrightarrow{wz}$ is an arc in $\overrightarrow{H}$. We know that $w \notin \{v = w_1, w_2, \ldots, w_d, u\}$, since $z \notin \{y_1, \ldots, y_d, v\}$. Let $H_z \in \mathcal{H}$, containing vz and uw, be obtained from H by twisting uv and wz. Since uv was the unique edge of H colored by 2, either $\varphi(uw) = 2$ or $\varphi(vz) = 2$. Suppose $w \in Z$. Since the maximum monochromatic degree is d and $\varphi(uw_j) = 2$ for all $j \in [d]$, $\varphi(uw) \neq 2$, so $\varphi(vz) = 2$. Suppose $w \in X$. Since $\varphi(wz) \neq 2$ and φ is X-constant, $\varphi(wz) = \varphi(uw) \neq 2$, so $\varphi(vz) = 2$. In both cases, $\varphi(vz) = 2$. Therefore, v is a $(1,2)$-max-vertex and the proof of (2) is done.

If $X = \emptyset$ then both (3) and (4) hold. So, assume that $X \neq \emptyset$. Let $H \in \mathcal{H}$. Assume that there in an edge of H with one endpoint in X and another in Z. Then there exist $x \in X$ and $y \in Z$ such that $\overrightarrow{yx}$ is an arc in $\overrightarrow{H}$. We know $y \notin \{v = w_1, \ldots, w_d, u\}$, because the successor of y in $\overrightarrow{H}$ is in X. If we twist yx and uv we get $H_x \in \mathcal{H}$ containing uy and vx, where one of these edges must have color 2. However, since $\varphi(xv) = \varphi(xy) \neq 2$, we must have $\varphi(yu) = 2$, and u has monochromatic degree $d+1$ in G, a contradiction. Hence there is no edge in $\overrightarrow{H}$ with one endpoint in X and another in Z, and thus X induces a 2-factor in H. In particular, since $Z \neq \emptyset$, H is not a Hamiltonian cycle, and $\mathcal{H} = \mathrm{F}_2$. Let $p, q \in X$ with $pq \in E(H)$. Since both $(H \setminus \{uv, pq\}) \cup \{pv, qu\}$ and $(H \setminus \{uv, pq\}) \cup \{pu, qv\}$ are 2-factors in $\mathcal{H}$, and φ is X-constant, either $\varphi(pv) = \varphi(pu) = 2$ or $\varphi(qv) = \varphi(qu) = 2$. In fact, since φ is X-constant, for each edge pq of H, where $p, q \in X$, all the edges from either p or q into Z have color 2. Since $H[X]$ is a union of cycles, at least half the edges between X and Z have color 2. This proves (3) and (4) and finishes the proof of Claim 1. □

Claim 2. The graph G does not contain a $(1, 2)$-max-vertex, a $(2, 3)$-max-vertex, and a $(3, 1)$-max-vertex at the same time.

Proof. Let x, y, and z be a $(1, 2)$-max-vertex, a $(2, 3)$-max-vertex, and a $(3, 1)$-max-vertex, respectively. Applying Claim 1 to $\{v, u\} = \{x, y\}$, then $\{v, u\} = \{y, z\}$, and then with $\{v, u\} = \{z, x\}$, the conclusion (4) gives that at least half of the edges between X and Z have color 2, 3, and 1, respectively. Since colors 1, 2, and 3 are distinct, we conclude $X = \emptyset$. Let $H \in \mathcal{H}$. Observe that x, y, and z could be incident only with edges of H with colors in $\{1, 2, 3\}$ in φ, so all other colors in H come from edges not incident with these vertices.

Let $\varphi^\star$ be obtained from φ by the following modification

$$c^\star(uv) = \begin{cases} 1 & u = x \text{ and } v \neq y, \\ 2 & u = y \text{ and } v \neq z, \\ 3 & u = z \text{ and } v \neq x, \\ \varphi(uv) & \text{otherwise.} \end{cases}$$

Observe that the union of edges of H with at least one endpoint in $\{x, y, z\}$ contains all colors $\{1, 2, 3\}$ in $\varphi^\star$. Hence H is polychromatic in $\varphi^\star$ and $\varphi^\star$ is $\mathcal{H}$-polychromatic. Moreover, $\varphi^\star$ is $\{x, y, z\}$-constant and all the other properties of (B) hold, which is a contradiction. This finishes the proof of Claim 2. □

Claim 3. If v is a $(1, 2)$-max-vertex and $u \in Z$ such that $\varphi(uv) = 2$, then u is a $(2, 1)$-max-vertex.

Proof. Let v be a $(1, 2)$-max-vertex and $u \in Z$ such that $\varphi(uv) = 2$. Claim 1 implies that u is a $(2, \star)$-max-vertex. Suppose for contradiction that u is a $(2, 3)$-max-vertex. Since the number of edges incident to v colored 2 is the same as the number of edges incident to u colored 3 and $\varphi(uv) = 2$, there is a vertex x such that $\varphi(ux) = 3$ and $\varphi(vx) = 1$. Again by Claim 1, x is a $(3, 1)$-max-vertex, contradicting Claim 2. □

Claim 4. If there is a $(1, 2)$-max-vertex, then there is no $(1, b)$-max-vertex for any $b \neq 2$.

Proof. By symmetry suppose for contradiction that $v \in Z$ is a $(1, 2)$-max-vertex and $u \in Z$ is a $(1, 3)$-max-vertex. Let $x \in Z$ be a vertex with $\varphi(vx) = 2$. By Claim 3, x is a $(2, 1)$-max-vertex. Notice $\varphi(ux) \in \{1, 2\} \cap \{1, 3\} = \{1\}$. Now Claim 3 applied to x and u gives that u is a $(1, 2)$-max-vertex, which is a contradiction. □

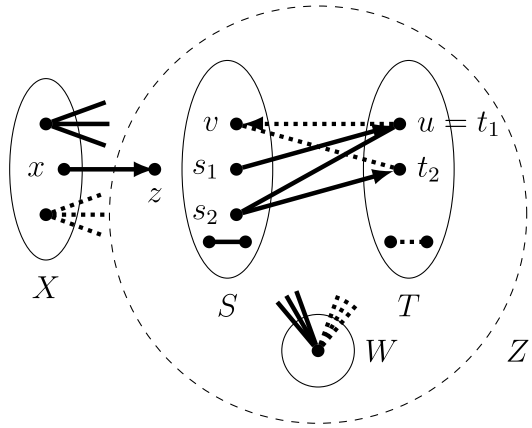

Figure 6: Final part of proof of Lemma 5 with only $(1, 2)$- and $(2, 1)$-max-vertices, solid edges correspond to color 1, dotted edges correspond to color 2.

Claims 3 and 4 imply that if there is an (a, b)-max-vertex, then $\{a, b\} = \{1, 2\}$.

Let S be the set of all $(1, 2)$-max-vertices, T be the set of all $(2, 1)$-max-vertices, and $W = Z \setminus (S \cup T)$. By Claim 3, both S and T are not empty. Edges within S and from S to W must have color 1 (because any minority color edge at a max-vertex is incident to a max-vertex of that color), edges within T and from T to W must have color 2, and each edge between S and T must have color 1 or 2.

Suppose $X = \emptyset$. Let $s \in S$ and $t \in T$. Let $\varphi^\star$ be obtained from φ by recoloring all edges incident to s to 1 and by recoloring all edges incident to t and not incident to s to 2. Notice that $\varphi^\star$ is $\mathcal{H}$-polychromatic. This contradicts the maximality of the monochromatic degree in φ. Therefore, $X \neq \emptyset$.

By symmetry, we can assume $|S| \leq |T|$. Let $v \in S$ and $u \in T$ be such that $\varphi(vu) = 2$. By the maximality of the monochromatic degree of v in Z, there exists $H \in \mathcal{H}$, where uv is the unique edge colored by 2. Recall $\ell = m - 1 - d$. Let $s_1, \ldots, s_\ell \in S$, where $\varphi(us_i) = 1$ for all $i \in [\ell]$. Let $\overrightarrow{H}$ be directed cycle(s) obtained by orienting edges of H such that $\overrightarrow{uv} \in \overrightarrow{H}$. Let t_i be a vertex such that $\overrightarrow{s_i t_i} \in \overrightarrow{H}$ for all $i \in [\ell]$. See Figure 6.

If there is an $i \in [\ell]$ such that $\varphi(vt_i) \neq 2$, then twist of of vu and $s_i t_i$ contains vt_i and us_i and leaves no edge colored 2, which is a contradiction with φ being $\mathcal{H}$-polychromatic. Hence $\varphi(vt_i) = 2$ and $t_i \in T$ for all $i \in [\ell]$. Notice v has exactly ℓ incident edges colored 2 and the other ends of these edges must be $t_1, \ldots, t_\ell$. By symmetry, we assume $t_1 = u$.

Suppose there exists $\overrightarrow{xz}$ in $\overrightarrow{H}$ with $x \in X$ and $z \in Z$. Since uv is the unique edge of $\overrightarrow{H}$ colored 2, $\varphi(xz)$ is not 2 and since φ is X-constant, $\varphi(xz) = \varphi(xu) \neq 2$. Notice that $z \notin \{u = t_1, \ldots, t_\ell\}$ since for every $i \in [\ell]$, the predecessor of t_i in $\overrightarrow{H}$ is s_i and $s_i \in S$. Hence $\varphi(vz) = 1$ and the twist of xz and uv contains xu and vz. Since $\varphi(xu) \neq 2$ and $\varphi(vz) \neq 2$, we get a contradiction to φ being $\mathcal{H}$-polychromatic. Therefore, there is no edge of H between

X and Z.

Since there is no edge of H between X and Z and $X \neq \emptyset$, H is not connected. Therefore, $\mathcal{H} = \mathrm{F}_2$.

Recall that all edges between T and W have color 2, hence they are not in H. Since there are no edges of H between X and Z, and all edges within T have color 2, every vertex in T has both neighbors from H in S. On the other hand, every vertex in S has at most two neighbors from H in T. Thus $|S| \geq |T|$. Recall we assumed $|S| \leq |T|$. Hence $|S| = |T|$ and there are no edges of H between $S \cup T$ and W.

Consider a bipartite graph B with vertex set $S \cup T$, edges st, $s \in S$, $t \in T$, and $\varphi(st) = 1$. Since vertices in T are $(2,1)$-max-vertices, each of them has degree exactly ℓ in B. Similarly, since vertices in S are $(1,2)$-max-vertices, each of them is not adjacent to exactly ℓ vertices of T. Therefore, all vertices in S have the same degree in B. Since $|S| = |T|$, we conclude B is an ℓ-regular graph.

If $\ell \geq 2$ then there exists a 2-factor K in B. Let $H^\star$ be obtained from H by removing edges incident to vertices in $S \cup T$ and adding K. Since all edges of K have color 1 and uv was the unique edge of H colored 2, we conclude $H^\star$ has no edge colored 2, contradicting the assumption that φ is $\mathcal{H}$-polychromatic.

Finally, if $\ell = 1$, then B is a matching on 4 vertices and the other two edges between S and T must have color 2. Hence $S \cup T$ does not contain a 2-factor in which uv would be the unique edge colored 2. This contradicts the existence of H.

This finishes the proof of Lemma 5. □

Proof of Theorem 2. Let $\mathcal{H} \in \{\mathrm{F}_2, \mathrm{HC}\}$. Let φ_1 be an optimal $\mathcal{H}$-polychromatic coloring of $E(K_n)$ with k colors and $[k]$ be the set of colors. We choose $X = \emptyset$, then we repeatedly apply Lemma 5. In the first application, we may get Lemma 5(B) and get $X = \{x, y, z\}$ that are unitary of distinct colors or Lemma 5(A) and $|X| = 1$. But after that Lemma 5(A) always applies. Note that there are no unitary vertices except possibly x, y, and z because each other vertex is incident to distinct colors c_x, c_y, c_z that are main colors of x, y, and z. This results in a combed edge-coloring φ with zero or three first unitary vertices and all others being ordered vertices.

Let M_i be the inherited color classes obtained from φ. Let $M_1, \ldots, M_{k-3}$ be the inherited color classes not containing x, y, or z. By Lemma 4, for each color class M_t there is j_t such that $|M_t(j_t)| \geq \frac{j_t}{2}$. By symmetry, assume $j_i < j_t$ for all $1 \leq i < t \leq k-3$. This leads to

$$|M_t(j_t)| \geq \sum_{i<t} |M_i(j_t)| \geq \sum_{i<t} |M_i(j_i)|$$

for $t = 2, \ldots, k-3$ and $|M_1| \geq 1$. Hence by induction we get

$$|M_t(j_t)| \geq 1 + \sum_{2 \leq i < t} |M_i(j_i)| \geq 1 + \sum_{2 \leq i < t} 2^{i-2} = 2^{t-2}.$$

Therefore, $|M_t| \geq 2^{t-2}$ for $t \geq 2$ and

$$n \geq \sum_{1 \leq t \leq k-3} |M_t| \geq 1 + \sum_{2 \leq t \leq k-3} 2^{t-2} \geq 2^{k-4}.$$

Hence $k \leq \log_2 n + 4$. By splitting the cases to (A) and (B), we could show $k \leq \log_2 n + 2$.

The lower bounds in Theorem 2 follow from colorings φ_{F_2} and φ_{HC}. Since every Hamiltonian cycle is also a 2-factor, we obtain $\mathrm{poly}_{\mathrm{F}_2}(K_n) \leq \mathrm{poly}_{\mathrm{HC}}(K_n)$. □

6 Closing Remarks

We show above that c from Theorem 2 is at most 4. It is possible to get a more precise version of Lemma 4 and use it to get sharp bounds in Theorem 2. We do not provide the details in order to keep the paper short and less technical. Details should be in the follow-up paper [8] together with generalizations which allow $\mathcal{H}$ to be the family of all 1-regular or 2-regular graphs that span all but a fixed number of vertices.

7 Acknowledgements

This research was initiated at a workshop made possible by the Alliance for Building Faculty Diversity in the Mathematical Sciences (DMS 0946431).

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