

Power propagation time and lower bounds for power domination number

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Abstract

We present a counterexample to a lower bound for power domination number given in Liao, Power domination with bounded time constraints, *J. Comb. Optim.*, in press 2014. We also define the power propagation time and make connections between the power domination propagation ideas in Liao and the (zero forcing) propagation time in Hogben et al, Propagation time for zero forcing on a graph, *Discrete Appl. Math.*, 2012.

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1 Introduction

Recent work has established a close connection between the power domination process on a graph and the zero forcing process on a graph. Notably, for a graph $G = (V(G), E(G))$ with zero forcing number $Z(G)$, power domination number $\gamma_P(G)$, and maximum degree $\Delta(G)$, it has been shown in [3] and [4] that

$$\gamma_P(G) \geq \left\lceil \frac{Z(G)}{\Delta(G)} \right\rceil. \quad (1)$$

Hence, it is not surprising that there are significant connections between the propagating processes for power domination and zero forcing. In this article, we extend this connection by considering the explicit parallels between power domination, zero forcing, and their corresponding propagation times. As above, our focus will be on establishing lower bounds for the power domination number, because for a graph or family of graphs, it is often easy to exhibit a power dominating set. However, showing this set has minimum cardinality among power dominating sets can be challenging.

A lower bound on power domination number using the propagation time of a particular power dominating set is presented in [6, Theorem 3]; using our notation this bound is

$$|S| \geq \frac{|V|}{\text{ppt}(G, S) \cdot \Delta(G) + 1} \quad (2)$$

for any power dominating set S with power propagation time $\text{ppt}(G, S)$, which is defined more precisely in Definition 1.1. To make this expression useful as a lower bound for $\gamma_P(G)$, an upper bound for $\text{ppt}(G, S)$ must be found. It is incorrectly claimed in the proof of [6, Theorem 3] that $\ell = \text{ppt}(G, S) \leq \text{diam}(G)$ for every power dominating set S , which would yield the following the incorrect lower bound for the power domination number

$$\frac{|V|}{\text{diam}(G) \cdot \Delta(G) + 1} \quad (3)$$

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33 where $\text{diam}(G)$ is the diameter of G (maximum distance between any two vertices).

34 This paper makes several contributions. In Section 2, we construct an infinite family of graphs
 35 having power domination number equal to 2 and having the value in (3) arbitrarily large (see Example 2.2),
 36 thereby showing that (3) is not a lower bound for power domination number. In Section 3, we discuss time-
 37 constrained or ℓ -round power domination using another interpretation of [6, Theorem 3] (Example 2.2 is
 38 still a counterexample to the diameter bound in [6, Theorem 3]). In Section 4, we present additional results
 39 about power propagation time. In particular, for trees we establish a lower bound for power domination
 40 number that is better than (3). In the remainder of this introduction, we give definitions of power domination
 41 number, zero forcing number, (power and zero forcing) propagation times, and related terms.

42 Here we reproduce the definition of the power domination process and number from [6] using our
 43 notation. The set of neighbors of vertex v is denoted by $N(v)$, and $N(S) = \bigcup_{v \in S} N(v)$. For a set S of
 44 vertices in a graph G , define the following sets:

- 45 1. $S^{[1]} = N[S] = S \cup N(S)$.
- 46 2. For $i \geq 1$, $S^{[i+1]} = S^{[i]} \cup \{w : \exists v \in S^{[i]}, N(v) \cap (V(G) \setminus S^{[i]}) = \{w\}\}$.

47 A set $S \subseteq V(G)$ is a *power dominating set* of a graph G if there is some ℓ such that $S^{[\ell]} = V(G)$. A *minimum*
 48 *power dominating set* is a power dominating set of minimum cardinality, and the *power domination number*
 49 of G , denoted by $\gamma_P(G)$, is the cardinality of a minimum power dominating set.

50 The zero forcing process is a coloring game on a graph. If u is a blue vertex and exactly one neighbor
 51 w of u is white, then change the color of w to blue; this is called the *color change rule* and we say that u
 52 *forces* w . A *zero forcing set* for G is a subset of vertices B such that if initially the vertices in B are colored
 53 blue and the remaining vertices are colored white, then repeated application of the color change rule can
 54 color all vertices of G blue. A *minimum zero forcing set* is a zero forcing set of minimum cardinality, and
 55 the *zero forcing number* $Z(G)$ of G is the cardinality of a minimum zero forcing set [2].

56 Next we connect ideas in [6] about the time ℓ needed to power dominate the entire graph to related
 57 results in the literature such as propagation time for zero forcing. For a graph $G = (V(G), E(G))$ and power
 58 dominating set S , define $S^{(0)} = S$, $S^{(1)} = N(S)$, and for $i \geq 1$,

$$59 \quad S^{(i+1)} = \{w : \exists v \in S^{(i)}, N(v) \cap (V(G) \setminus S^{(i)}) = \{w\}\}.$$

60 The computation of the sets $S^{(i)}$ and $S^{[i]}$ for a specific graph is illustrated in Example 4.5 below. This
 61 double notation $S^{(i)}$ and $S^{[i]}$ accommodates both the usage in [6], with $S^i = S^{[i]}$, and the notation used for
 62 the sets of blue vertices in the definition of (zero forcing) propagation time in [5], where $B = B^{(0)}$ is the set
 63 of vertices initially colored blue and

$$64 \quad B^{(i+1)} = \{w : \exists v \in B^{(i)}, N(v) \cap (V(G) \setminus \bigcup_{j=0}^i B^{(j)}) = \{w\}\}.$$

65 Then for a minimum zero forcing set B , $\text{pt}(G, B)$ is the least integer t such that $V(G) = \bigcup_{i=0}^t B^{(i)}$, or
 66 equivalently, the least t such that $B^{(t+1)} = \emptyset$. The *propagation time* of G , $\text{pt}(G)$ is the minimum of $\text{pt}(G, B)$
 67 over all minimum zero forcing sets B of G [5].

68 We define the power propagation time of a power dominating set S and of a graph G in analogy with
 69 the definitions of propagation time $\text{pt}(G, B)$ and $\text{pt}(G)$ for zero forcing.

70 **Definition 1.1.** For a power dominating set S , the *power propagation time* of S in G , denoted here by
 71 $\text{ppt}(G, S)$ and referred to as ℓ in [6], is the least integer ℓ such that $S^{[\ell]} = V(G)$, or equivalently, the least ℓ
 72 such that $S^{[\ell+1]} = S^{[\ell]}$. The *power propagation time* of G is

$$73 \quad \text{ppt}(G) = \min\{\text{ppt}(G, S) \mid S \text{ is a minimum power dominating set of } G\}.$$

74 **Remark 1.2.** Since S is a power dominating set if and only if $N(S)$ is a zero forcing set for $G - S$, and
 75 after the domination step the power propagation process for S on G is the zero forcing process for $N(S)$ on
 76 $G - S$ [3], $\text{ppt}(G, S) = \text{pt}(G - S, N(S)) + 1$.

2 Lower bounds for power dominating number

Unfortunately there is an error in Theorem 3 in [6], which states:

Given a connected graph $G = (V, E)$ and a positive integer time constraint ℓ , we can derive the lower bound of the minimum cardinality of a PDS in G . $\gamma_P(G) \geq \frac{|V|}{\ell \cdot \Delta + 1} \geq \frac{|V|}{\text{diam}_G \cdot \Delta + 1}$, where Δ is the maximum degree of G and diam_G is the diameter of G .

The first of these two inequalities is correct if ℓ is the power propagation time of a minimum power dominating set. Using our power propagation time terminology, the proof of [6, Theorem 3] shows that

$$|S| \geq \frac{|V(G)|}{\text{ppt}(G, S) \cdot \Delta(G) + 1}.$$

By choosing a minimum power dominating set having minimum propagating time, we optimize this inequality (again restated in our notation):

Theorem 2.1. [6, Theorem 3] For a connected graph $G = (V, E)$,

$$\gamma_P(G) \geq \frac{|V|}{\text{ppt}(G) \cdot \Delta(G) + 1}.$$

The error in the proof occurs in trying to compare the non-comparable parameters $\text{ppt}(G)$ (or $\text{ppt}(G, S)$) and $\text{diam}(G)$. The last sentence of the proof (on p. 15) reads: *Because ℓ is not larger than the diameter of the given graph G , ...*. This statement is not correct in general (although it is true for trees, as shown in Theorem 4.2 below). It is easy to construct examples where $\ell = \text{ppt}(G, S)$ is greater than the diameter. But more importantly, the diameter “lower bound” in [6, Theorem 3], i.e., $\gamma_P(G) \geq \frac{|V(G)|}{\text{diam}(G) \cdot \Delta + 1}$, is not correct, as the next example shows.

Example 2.2. Given an integer $\Delta \geq 3$, construct the graph H_Δ with three levels of vertices as follows:

1. The first vertex is numbered 1. This one vertex is on level 1.
2. Vertex 1 has Δ neighbors, numbered $2, \dots, \Delta + 1$. These vertices are on level 2.
3. Each level 2 vertex has $\Delta - 1$ neighbors on level 3. This adds $\Delta(\Delta - 1) = \Delta^2 - \Delta$ vertices, numbered $\Delta + 2, \dots, \Delta^2 + 1$.
4. Add edges to make a path along level 3, so vertex i is adjacent to vertex $i + 1$ for $\Delta + 2 \leq i \leq \Delta^2$.

Figure 1 shows the graph H_Δ for $\Delta = 9$ (the three levels are the three rows of vertices).

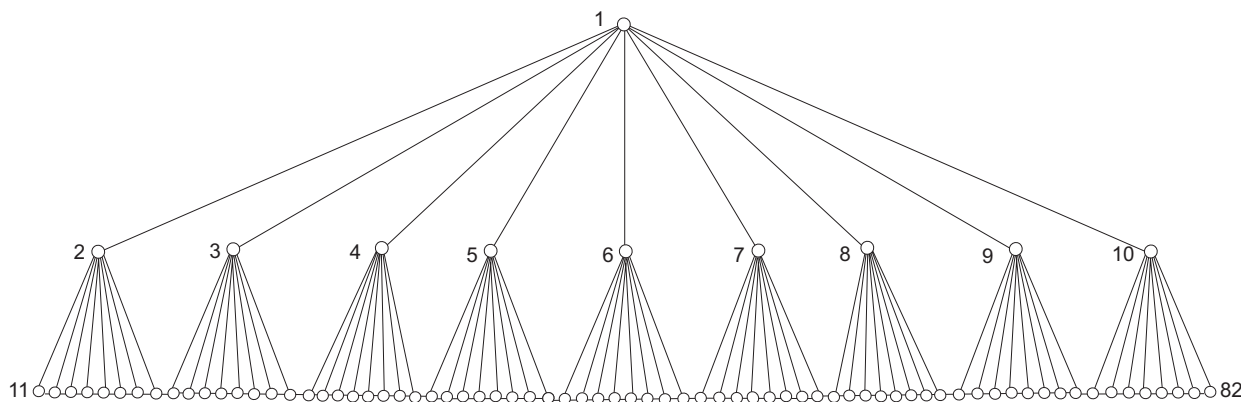


Figure 1: The graph H_Δ for $\Delta = 9$

Then $|V(H_\Delta)| = 1 + \Delta + \Delta(\Delta - 1) = 1 + \Delta^2$, $\text{diam}(H_\Delta) = 4$, and the maximum degree is $\Delta(H_\Delta) = \Delta$. The power domination number is $\gamma_P(H_\Delta) = 2$, because no one vertex power dominates H_Δ , and vertices 1 and $\Delta + 2$ are a power dominating set. But

$$\frac{|V(H_\Delta)|}{\Delta(H_\Delta) \text{diam}(H_\Delta) + 1} = \frac{\Delta^2 + 1}{4\Delta + 1} \approx \frac{\Delta}{4}.$$

106 By choosing Δ large, $\frac{|V(H_\Delta)|}{\Delta(H_\Delta) \text{diam}(H_\Delta) + 1}$ can be made arbitrarily large. In particular, for $\Delta = 9$,

$$107 \quad \frac{|V(H_\Delta)|}{\Delta(H_\Delta) \text{diam}(H_\Delta) + 1} = \frac{82}{37} > 2 = \gamma_P(H_\Delta).$$

108 While the relationship in Theorem 2.1 is correct, $\frac{|V(G)|}{\text{ppt}(G) \cdot \Delta(G) + 1}$ is not useful as a lower bound for
 109 $\gamma_P(G)$, because one must know $\gamma_P(G)$ in order to compute $\text{ppt}(G)$. One can, however, use the relationship
 110 in Theorem 2.1 to obtain a lower bound on power propagation time, assuming one knows $\gamma_P(G)$):

111 **Proposition 2.3.** *For a graph G ,*

$$112 \quad \text{ppt}(G) \geq \left\lceil \frac{|V(G)| - \gamma_P(G)}{\gamma_P(G) \cdot \Delta(G)} \right\rceil.$$

113 3 ℓ -round power domination number

114 Note that the use of ℓ in [6, Theorem 3] is open to interpretation, because it is used in at least two ways in
 115 [6]:

- 116 1. As what we call the (minimum) power propagation time of G and denote by $\text{ppt}(G)$.
- 117 2. As a time constraint, as in ℓ -round power domination, meaning minimum number of vertices needed
 118 to power-dominate G in at most ℓ rounds (see also [1]).

119 Although ℓ is referred to as a ‘time constraint’ in Theorem 3 of [6], to salvage any correct result from the
 120 proof it seems necessary to assume that the intended meaning is either the minimum power propagation time
 121 (as we have done in the preceding discussion), or that the symbol $\gamma_P(G)$ is being used to mean an ℓ -round
 122 power domination number, which should more properly be denoted by $\gamma_P^\ell(G)$ or similar notation. Clearly
 123 $\gamma_P^\ell(G) \geq \gamma_P(G)$. With the second interpretation and the notation $\gamma_P^\ell(G)$, we obtain the following alternate
 124 (and also correct) version of part of [6, Theorem 3]:

125 **Theorem 3.1.** [6, Theorem 3] *For a connected graph $G = (V, E)$ and any positive integer time constraint ℓ ,*

$$126 \quad \gamma_P^\ell(G) \geq \frac{|V|}{\ell \cdot \Delta(G) + 1}.$$

127 But again the second inequality (involving the diameter) is incorrect, as can be seen from Example
 128 2.2 with ℓ chosen large enough.

129 4 Additional properties of power propagation time

130 In Section 2 we showed that there are graphs G for which $\text{ppt}(G) > \text{diam}(G)$. However (as is the case for
 131 propagation time [5, Theorem 4.3]), $\text{ppt}(T) \leq \text{diam}(T)$ when T is a tree (in fact this can be strengthened
 132 when T is not K_2 – see Theorem 4.2).

133 First we establish a relationship between power propagation time and maximum trail length. A *walk*
 134 $v_0 v_1 \cdots v_p$ in G is a subgraph with vertex set $\{v_0, v_1, \dots, v_p\}$ and edge set $\{v_0 v_1, v_1 v_2, \dots, v_{p-1} v_p\}$ (vertices
 135 and/or edges may be repeated in a walk but duplicates are removed from sets). A *trail* is a walk with no
 136 repeated edges (vertices may be repeated). Of course, a path is a trail with no repeated vertices. The *length*
 137 of a trail $P = v_0 v_1 \cdots v_p$ is p , i.e., the number of edges in P .

138 The proofs of the analogous results for (zero forcing) propagation time in [5] can be adapted to prove
 139 $\text{ppt}(T) \leq \text{diam}(T)$, but we present a different proof using edge labeling by propagation time that produces
 140 a stronger result. Given a power dominating set S , the power propagation process naturally labels a vertex
 141 by the time at which it is observed, that is, $t(v) = i$ for $v \in S^{(i)}$. We can extend this labeling to edges
 142 by $t(uv) = \max\{t(u), t(v)\}$. A trail $P = v_0 v_1 \cdots v_p$ is called *monotone* if $p \geq 1$, $t(v_{i-1} v_i) \leq t(v_i v_{i+1}) \leq$
 143 $t(v_{i-1} v_i) + 1$ for $i = 1, \dots, p - 1$, and $t(v_{p-1} v_p) = t(v_p)$.

144 **Lemma 4.1.** *Let G be a graph and let S be a power dominating set of G with no vertices of degree 1.*
 145 *Then for every edge $e \in V(G)$, there is a monotone trail of length $t(e) + 1$ in which e is the last edge and*
 146 *$t(v_p) = t(e)$, where v_p is the last vertex of the trail.*

147 *Proof.* Let $e = uv$ with $u, v \in V(G)$. This proof is by induction on $t(e)$. If $t(e) = 0$, then $P = uv$ is the
 148 required trail. Suppose $t(e) = 1$. Then either $t(u) = t(v) = 1$ or $t(u) = 0$ and $t(v) = 1$ (or vice versa). If
 149 $t(u) = t(v) = 1$, then there is a $w \in S$ such that $u \in N(w)$, and $P = wuv$ is the required trail. So suppose
 150 $t(u) = 0$ and $t(v) = 1$. Because the degree of u is at least 2, there is an edge $e' \neq e$ that is incident to u and
 151 necessarily $t(e')$ is equal to 0 or 1, so $P = wuv$ is the required trail, where $e' = uv$. This proves the base
 152 cases.

153 Assume the statement is true when $t(e) < i$. Now assume that $t(e) = i \geq 2$. Then at least one of
 154 the vertices, say v , is forced at time i . Let $w \in V(G)$ be the vertex that forces v at time i . (Note: It is
 155 possible that $w = u$.) If w is forced at time $i - 1$, then w is incident to an edge e' such that $t(e') = i - 1$.
 156 Therefore, there is a monotone trail of length at least i that ends at w . By adjoining e and v to this trail,
 157 there is a monotone trail of length at least $i + 1$ that contains e as the last edge and v as the last vertex. If
 158 w is not forced at time $i - 1$, then there exists a vertex $w' \in N(w)$ such that $t(w') = i - 1$. So there is a
 159 monotone trail $P = v_0 \dots v_{p-1}w'$ of length at least i that ends at w' . Then $v_0 \dots v_{p-1}w'wv$ is a monotone
 160 trail of length at least $i + 1$ that has e as its last edge and $t(e) = t(v)$. \square

161 **Theorem 4.2.** *For every tree T with at least 3 vertices, $\text{ppt}(T) \leq \text{diam}(T) - 1$.*

162 *Proof.* Let S be a power dominating set of T such that $\text{ppt}(T, S) = \text{ppt}(T)$. Assume S contains a vertex v
 163 with degree 1. Let u be the vertex adjacent to v in T . The degree of u is at least 2, because T has at least
 164 3 vertices. Since S is minimum, $u \notin S$. Therefore $S' = S \setminus \{v\} \cup \{u\}$ is also a minimum power dominating
 165 set of T , and $\text{ppt}(T, S') \leq \text{ppt}(T, S)$. So T has a minimum power dominating set containing no vertices of
 166 degree 1 that has power propagation time of $\text{ppt}(T)$. By Lemma 4.1, T contains a monotone trail of length
 167 at least $\text{ppt}(T) + 1$, and every trail in a tree is a path. Therefore, T has a path of length at least $\text{ppt}(T) + 1$.
 168 So $\text{ppt}(T) + 1 \leq \text{diam}(T)$. \square

169 **Corollary 4.3.** *For every tree T with at least 3 vertices,*

$$170 \quad \gamma_P(T) \geq \left\lceil \frac{|V(G)|}{(\text{diam}(G) - 1) \cdot \Delta(G) + 1} \right\rceil.$$

171 A natural question is whether there is any relationship between the parameters power propagation
 172 time and propagation time. However, these parameters are non-comparable. It is easy to construct examples
 173 of graphs G with $\text{ppt}(G) < \text{pt}(G)$, for example a path on at least 3 vertices or a star on at least 3 vertices.
 174 A hypercube in dimension at least four has the reverse inequality, as the next example shows.

175 **Example 4.4.** The d th hypercube, Q_d , is defined recursively by $Q_1 = K_2$ and $Q_d = Q_{d-1} \square K_2$. It is known
 176 that $Z(Q_4) = 8$ [2] and $\text{pt}(Q_4) = 1$ [5]. As is observed in [4], $\gamma_P(Q_4) = 2$. Since $\Delta(Q_4) = 4$, a set of 2
 177 vertices can dominate at most 10 vertices, and $\text{ppt}(Q_4) \geq 2$ (Proposition 2.3 also gives 2 as a lower bound).

178 It was observed in [5, Example 1.10] that a gap can occur in the possible propagation times among
 179 minimum zero forcing sets, and the same situation arises for power propagation times of minimum power
 180 dominating sets.

181 **Example 4.5.** Let G be the graph shown in Figure 2. Since $S = \{c\}$ is a power dominating set, $\gamma_P(G) = 1$.
 182 Observe that $S^{(1)} = \{b, d, h\}$, $S^{(2)} = \{a, e, p\}$, $S^{(3)} = \{f, q\}$, and $S^{(4)} = \{g\}$, and $S^{[1]} = \{c, b, d, h\}$, $S^{[2]} =$
 183 $\{c, b, d, h, a, e, p\}$, $S^{[3]} = \{c, b, d, h, a, e, p, f, q\}$, and $S^{[4]} = \{c, b, d, h, a, e, p, f, q, g\}$. Thus $\text{ppt}(G, S) = 4$. The
 184 only other minimum power dominating set is $T = \{a\}$. Since $T^{[1]} = \{a, b, f, g\}$, $T^{[2]} = \{a, b, f, g, c, e\}$, $T^{[3]} =$
 185 $\{a, b, f, g, c, e, d\}$, $T^{[4]} = \{a, b, f, g, c, e, d, h\}$, $T^{[5]} = \{a, b, f, g, c, e, d, h, p\}$, $T^{[6]} = \{a, b, f, g, c, e, d, h, p, q\}$,
 186 $\text{ppt}(G, T) = 6$. Thus $\text{ppt}(G, S) = 4$, $\text{ppt}(G, T) = 6$, and there is no minimum power dominating set U
 187 having $\text{ppt}(G, U) = 5$.

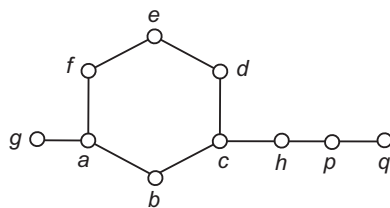


Figure 2: The graph G for Example 4.5

References

188

- 189 [1] A. Aazami. Domination in graphs with bounded propagation: algorithms, formulations and hardness
 190 results. *J. Comb. Optim.*, 19: 429–456, 2010
- 191 [2] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioaba, D.
 192 Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelsen, S. Narayan, O. Pryporova,
 193 I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, A. Wangsness). Zero forcing sets
 194 and the minimum rank of graphs. *Lin. Alg. Appl.*, 428: 1628–1648, 2008.
- 195 [3] K.F. Benson, D. Ferrero, M. Flagg, V. Furst, L. Hogben, V. Vasilevska, B. Wissman. Power domination
 196 and zero forcing. Under review. Available at <http://arxiv.org/abs/1510.02421>.
- 197 [4] N. Dean, A. Ilic, I. Ramirez, J. Shen, K. Tian. On the Power Dominating Sets of Hypercubes. *2011 IEEE*
 198 *14th International Conference on Computational Science and Engineering (CSE)*, IEEE Conference
 199 Publications, 488 – 491, 2011, DOI: 10.1109/CSE.2011.89
- 200 [5] L. Hogben, M. Huynh, N. Kingsley, S. Meyer, S. Walker, M. Young. Propagation time for zero forcing
 201 on a graph. *Discrete Appl. Math.*, 160: 1994–2005, 2012.
- 202 [6] C-S. Liao. Power domination with bounded time constraints. *J. Comb. Optim.*, DOI 10.1007/s10878-
 203 014-9785-2, 2014.