ON WEAK CHROMATIC POLYNOMIALS OF MIXED GRAPHS

MATTHIAS BECK, DANIEL BLADO, JOSEPH CRAWFORD, TAÏNA JEAN-LOUIS, AND MICHAEL YOUNG

Abstract. A *mixed graph* is a graph with directed edges, called *arcs*, and undirected edges. A k-coloring of the vertices is *proper* if colors from $\{1, 2, \ldots, k\}$ are assigned to each vertex such that u and v have different colors if uv is an edge, and the color of u is less than or equal to (resp. strictly less than) the color of v if uv is an arc. The *weak* (resp. *strong*) *chromatic polynomial* of a mixed graph counts the number of proper k-colorings. Using order polynomials of partially ordered sets, we establish a reciprocity theorem for weak chromatic polynomials giving interpretations of evaluations at negative integers.

1. Introduction

A *mixed graph* $G = (V, E, A)$ consists of a set of vertices, $V = V(G)$, a set of undirected edges, $E = E(G)$, and a set of directed edges, $A = A(G)$. For convenience, the elements of E will be called *edges* and the elements of A will be called *arcs*. Given adjacent vertices $u, v \in V$, an edge will be denoted by uv and an arc will be denoted by $\vec{uv}$.

A k*-coloring* of a mixed graph G is a mapping $c : V \to [k]$, where $[k] := \{1, 2, \ldots, k\}$. A *weak* (resp. *strong*) *proper k-coloring* of G is a k-coloring such that

$$c(u) \neq c(v) \quad \text{if } uv \in E \qquad \text{and} \qquad c(u) \leq c(v) \quad (\text{resp. } c(u) < c(v)) \quad \text{if } \vec{uv} \in A\,.$$

The *weak* (resp. *strong*) *chromatic polynomial*, denoted by $\chi_G(k)$ (resp. $\widehat{\chi}_G(k)$), is the number of weak (resp. strong) proper k-colorings of G. It is well known (see, e.g., [4]) that these counting functions are indeed polynomials in k. Coloring problems in mixed graphs have various applications, for example in scheduling problems in which one has both disjunctive and precedence constraints (see, e.g., [2, 3, 5]).

An *orientation* of a mixed graph G is obtained by orienting the edges of G, i.e., assigning one of u and v to be the head/tail of the edge $uv \in E$; if v is the head we use the notation $u \to v$. (An arc $\vec{uv}$, for which we also use the notation $u \to v$, cannot be re-oriented.) An orientation of a mixed graph is *acyclic* if it does not contain any directed cycles. A mixed graph is *acyclic* if all of its possible orientations are acyclic. A coloring c and an orientation of G are *compatible* if for every $u \to v$ in the orientation, $c(u) \leq c(v)$.

A famous theorem of Stanley says that, for any graph $G = (V, E, \varnothing)$ and positive integer k, $(-1)^{|V|}\chi_G(-k)$ enumerates the pairs of k-colorings and compatible acyclic orientations of G and, in particular, $(-1)^{|V|}\chi_G(-1)$ equals the number of acyclic orientations of G [7]; this is an example of

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a *combinatorial reciprocity theorem.* More recently, Beck, Bogart, and Pham proved the following analogue of Stanley's reciprocity theorem for the strong chromatic polynomial of a mixed graph [1]:

Theorem 1. *For any mixed graph $G = (V, E, A)$ and positive integer k, $(-1)^{|V|}\widehat{\chi}_G(-k)$ equals the number of k-colorings of G, each counted with multiplicity equal to the number of compatible acyclic orientations of G.*

In this paper, we complete the picture by proving a reciprocity theorem for *weak* chromatic polynomials $\chi_G(k)$ of mixed graphs. A coloring c and an orientation of G are *intercompatible* if for every $u \to v$ in the orientation,

$$c(u) \le c(v) \ \text{ if } uv \in E(G) \qquad \text{and} \qquad c(u) < c(v) \ \text{ if } \vec{uv} \in A(G)\,.$$

Our main results is:

Theorem 2. *For any acyclic mixed graph $G = (V, E, A)$ and positive integer k, $(-1)^{|V|}\chi_G(-k)$ equals the number of k-colorings of G, each counted with multiplicity equal to the number of intercompatible acyclic orientations of G.*

One can prove this theorem along somewhat similar lines to the (geometric) approach used in [1], though there are subtle details that distinguish the case of weak chromatic polynomials from the one of strong chromatic polynomials. For example, although both Theorems 1 and 2 result in relating k-colorings of a mixed graph to its acyclic orientations, the reciprocity theorem for strong chromatic polynomials applies to all mixed graphs G, while the reciprocity theorem for weak chromatic polynomials requires the condition that G be an acyclic mixed graph: without this condition, Theorem 2 is not necessarily true.

Our proof of Theorem 2 applies Stanley's reciprocity theorem for *order polynomials*, stated in Section 2, which also contains the proof of Theorem 2. In Section 3 we give a deletion–contraction method for computing the weak and strong chromatic polynomials for mixed graphs, as well as an example that shows Theorem 2 may not hold for mixed graphs that are not acyclic.

2. Posets, Order Polynomials, and the Proof of Theorem 2

Recall that a partially ordered set (a *poset*) is a set P with a relation $\preceq$ that is reflexive, antisymmetric, and transitive. Following [6] (see also [7, Chapter 3]), we define an *ω-labeling* of a poset with n elements as a bijection $\omega : P \to [n]$, and the *order polynomial* $\Omega_{P,\omega}(k)$ as

$$\Omega_{P,\omega}(k) := \#\left\{(x_1, x_2, ..., x_n) \in [k]^n : \begin{array}{l} x_u \le x_v \text{ if } u \preceq v \text{ and } \omega(u) < \omega(v) \\ x_u < x_v \text{ if } u \preceq v \text{ and } \omega(u) > \omega(v) \end{array}\right\}.$$

Stanley [6] proved that $\Omega_{P,\omega}(k)$ is indeed a polynomial in k. The *complementary labeling* to ω is the $\overline{\omega}$-labeling of P defined by $\overline{\omega}(v) := n + 1 - \omega(v)$. Thus

$$\Omega_{P,\overline{\omega}}(k) = \#\left\{(x_1, x_2, ..., x_n) \in [k]^n : \begin{array}{l} x_u < x_v \text{ if } u \preceq v \text{ and } \omega(u) < \omega(v) \\ x_u \le x_v \text{ if } u \preceq v \text{ and } \omega(u) > \omega(v) \end{array}\right\}.$$

Theorem 3 (Stanley [6])**.** $\Omega_{P,\omega}(-k) = (-1)^{|P|}\,\Omega_{P,\overline{\omega}}(k)\,.$

The reciprocity relation given in Theorem 3 takes on a special form when ω is a *natural labelling* of P, i.e., one that respects the order of P. (It is easy to see that every poset has a natural labelling.) In this case $\Omega_{P,\omega}(k)$ simply counts all order preserving maps $x : P \to [k]$ (i.e., $u \preceq v \Longrightarrow x_u \le x_v$), whereas $\Omega_{P,\overline{\omega}}(k)$ couns all *strictly* order preserving maps $x : P \to [k]$ (i.e., $u \prec v \Longrightarrow x_u < x_v$). Theorem 3 implies that these two counting functions are reciprocal.

Throughout the rest of this paper, a proper k-coloring will refer only to a *weak* proper k-coloring of a mixed graph, and a chromatic polynomial will refer only to a *weak* chromatic polynomial of a mixed graph. For a mixed graph $G = (V, E, A)$, the chromatic polynomial $\chi_G(k)$ can be written as

$$\chi_G(k) = \# \left\{ (x_1, \ldots, x_n) \in [k]^n : \begin{array}{l} x_u \leq x_v \text{ if } \vec{uv} \in A, \\ x_u \neq x_v \text{ if } uv \in E \end{array} \right\}.$$

Each acyclic orientation of G can be translated into a poset by letting $P = V(G)$ and introducing, for each $u \to v$ in the orientation, the relation $u \preceq v$.

Throughout the remainder of this section, we fix an acyclic mixed graph G and denote by $G_1, G_2, \ldots, G_m$ the (acyclic) orientations of G. For each $1 \leq i \leq m$, denote P_i as the poset created by the orientation G_i, and let $\phi_{G_i}(k)$ be the number of proper k-colorings of G_i that are also proper k-colorings of G.

Lemma 4. *If G is an acyclic mixed graph, then* $\chi_G(k) = \sum_{i=1}^{m} \phi_{G_i}(k)\,.$

Proof. It is clear that each proper k-coloring of G is a proper k-coloring of G_i for some $1 \leq i \leq m$. Conversely, assuming $E(G) \neq \varnothing$, for any $1 \leq i < j \leq k$, there is some $uv \in E(G)$ such that $u \to v$ in G_i and $v \to u$ in G_j. This implies that there is no coloring that is a proper k-coloring of G_i and G_j. If $E(G) = \varnothing$ then G is the only orientation of itself. □

Lemma 5. *For each G_i, there exists an ω_i-labeling of P_i such that*

$$\phi_{G_i}(k) = \Omega_{P_i,\omega_i}(k)\,.$$

Moreover, $\Omega_{P_i,\overline{\omega}_i}(k)$ is the number of k-colorings intercompatible with G_i.

Proof. Given the orientation G_i, let R_i be the orientation of G obtained by reversing the orientation of the edges in G_i (but not the arcs). We will construct ω_i recursively.

Since R_i is acyclic, there exists a vertex $v \in V$ such that all edges and arcs incident to v are oriented away from it. Set $\omega_i(v) := 1$ and remove v and the arcs incident to v. Since R_i is acyclic, $R_i - v$ must also be acyclic. Now repeat, assigning each vertex in the process consecutive ω_i-labels. This gives ω_i-labels that satisfy

$$\omega_i(u) < \omega_i(v) \qquad \Longrightarrow \qquad u \to v \text{ in } R_i\,,$$

resulting in an ω_i-labeling of P_i, the poset corresponding to G_i, that satisfies for $u \preceq v$

$$\begin{aligned} \omega_i(u) < \omega_i(v) \qquad &\Longrightarrow \qquad \vec{uv} \in A(G) \\ \omega_i(u) > \omega_i(v) \qquad &\Longrightarrow \qquad uv \in E(G)\,. \end{aligned}$$

So

$$\begin{aligned} \Omega_{P_i,\omega_i}(k) &= \# \left\{ (x_1, x_2, ..., x_n) \in [k]^n : \begin{array}{l} x_u \leq x_v \text{ if } u \preceq v \text{ and } \omega_i(u) < \omega_i(v) \\ x_u < x_v \text{ if } u \preceq v \text{ and } \omega_i(u) > \omega_i(v) \end{array} \right\} \\ &= \# \left\{ (x_1, x_2, ..., x_n) \in [k]^n : \begin{array}{l} x_u \leq x_v \text{ if } u \to v \text{ in } G_i \text{ and } \vec{uv} \in A(G) \\ x_u < x_v \text{ if } u \to v \text{ in } G_i \text{ and } uv \in E(G) \end{array} \right\} \\ &= \phi_{G_i}(k)\,. \end{aligned}$$

For the second part of the proof, recall that

$$\begin{aligned}\Omega_{P_i,\overline{\omega_i}}(k) &= \#\left\{(x_1,x_2,...,x_n)\in[k]^n : \begin{array}{l} x_u < x_v \text{ if } u \preceq v \text{ and } \omega_i(u) < \omega_i(v) \\ x_u \le x_v \text{ if } u \preceq v \text{ and } \omega_i(u) > \omega_i(v)\end{array}\right\} \\ &= \#\left\{(x_1,x_2,...,x_n)\in[k]^n : \begin{array}{l} x_u < x_v \text{ if } u \preceq v \text{ and } \vec{uv} \in A(G) \\ x_u \le x_v \text{ if } u \preceq v \text{ and } uv \in E(G)\end{array}\right\} \\ &= \# \text{ colorings intercompatible with } G_i\,.\end{aligned}$$

□

Proof of Theorem 2. If G is an acyclic mixed graph, then by Lemma 4,

$$\begin{aligned}\chi_G(-k) &= \sum_{i=1}^m \phi_{G_i}(-k) \\ &= \sum_{i=1}^m \Omega_{P_i,\omega_i}(-k) \quad \text{(by Lemma 5)} \\ &= \sum_{i=1}^m (-1)^{|P_i|}\Omega_{P_i,\overline{\omega_i}}(k) \quad \text{(by Theorem 3)} \\ &= (-1)^{|V|}\sum_{i=1}^m \Omega_{P_i,\overline{\omega_i}}(k)\,.\end{aligned}$$

By applying Lemma 5 again, the proof is completed. □

3. Deletion–Contraction Computations

Let $G=(V,E,A)$ be a mixed graph, $e\in E(G)$, and $a\in A(G)$. Define $G-e=(V,E-e,A)$ as the mixed graph with edge e deleted and $G-a=(V,E,A-a)$ as the mixed graph with arc a deleted. An edge or arc is *contracted* by deleting the edge or arc and identifying the vertices incident to it (keeping only one copy of each edge and arc). Denote G/e as the mixed graph obtained by contracting edge e in G and G/a as the mixed graph obtained by contracting arc a in G. The standard proof for the deletion–contraction formula for (unmixed) graphs gives:

Proposition 6. *If G is a mixed graph and $e\in E(G)$, then*

$$\chi_G(k)=\chi_{G-e}(k)-\chi_{G/e}(k)\,.$$

Define G_a as the mixed graph G with arc a directed in the reverse direction. In other words, if $a=\vec{uv}$ then $G_a=(V,E,A-\{\vec{uv}\}\cup\{\vec{vu}\})$.

Proposition 7. *If G is a mixed graph and $a\in A(G)$, then*

$$\chi_G(k)+\chi_{G_a}(k)=\chi_{G-a}(k)+\chi_{G/a}(k)\,.$$

Proof. Let $a=\vec{uv}$, C be the set of proper k-colorings of G, and C_a be the set of proper k-colorings of G_a. Therefore, $C\cap C_a$ is the set of proper k-colorings of G such that $c(u)=c(v)$ and $(C\cup C_a)-(C\cap C_a)$ is the set of proper k-colorings of G and G_a such that $c(u)\neq c(v)$.

If $c\in(C\cup C_a)-(C\cap C_a)$, then c is a proper k-coloring of $G-a$. If $c\in C\cup C_a$, then c is a k-coloring of $G-a$ and corresponds to a proper k-coloring of G/a in which the vertex created by identifying u and v is colored with $c(u)$. Conversely, each proper k-coloring of $G-a$ and G/a corresponds to a unique k-coloring of G or G_a. □

Propositions 6 and 7 give the following equations:

$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k) \tag{1}$$

$$\chi_G(k) = \chi_{G-a}(k) + \chi_{G/a}(k) - \chi_{G_a}(k)\,. \tag{2}$$

Equation (1) is very useful in computing the chromatic polynomials since it recursively gives the chromatic polynomial of a mixed graph as a difference of (in the number of vertices or edges) smaller mixed graphs. On the other hand, equation (2) gives the chromatic polynomial of G in terms of G_a, which is not a smaller graph. However, we will show how it can be used in computation.

A directed graph $G = (V, \varnothing, A)$ is *strongly connected* if for any pair of vertices $u, v \in V$ there exists a directed path from u to v.

Proposition 8. *If G is a strongly connected directed graph, then $\chi_G(k) = k$.*

Proof. Fix $u \in V$, and let c be a proper coloring of G. Since there is a directed path from u to any v and vice versa, $c(v) \le c(u) \le c(v)$ for every $v \in V$. Therefore, $c(u) = c(v)$ for every $v \in V$, and since there are k colors that can be assigned to u, $\chi_G(k) = k$. □

Given a subgraph S of G, denote G/S as the mixed graph G with all edges and arcs of S removed and all vertices of S identified to one vertex.

Proposition 9. *Let G be a mixed graph and S be a strongly connected directed subgraph of G. Then $\chi_G(k) = \chi_{G/S}(k)$.*

Proof. Let s be the vertex that S contracts to in G/S. For each proper k-coloring of G, the vertices of S must all be colored the same color j. By defining $c(s) = j$ we get a bijection between the proper k-colorings of G and G/S. □

Computing the chromatic polynomial of a mixed graph is reduced to computing the chromatic polynomial of smaller directed graphs by applying Proposition 6. Computing the chromatic polynomial of a directed graph is reduced to computing the chromatic polynomial of smaller acyclic directed graphs (directed trees) by recursively reversing arcs and applying Proposition 7 until a strongly connected subgraph is created and Proposition 9 can be applied. Note that a strongly connected subgraph of a directed graph can be obtained by reversing arcs as long as the underlying graph is not acyclic.

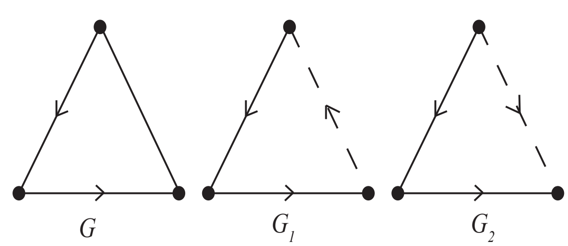

FIGURE 1. A mixed graph G and its two orientations.

As an example, let $G = (\{u, v, w\}, \{uv\}, \{\vec{vw}, \vec{wu}\})$ (shown in Figure 1). G is a cyclic mixed graph since it has an orientation, G_1, that contains a directed cycle. Consider $k = 2$. If c is an intercompatible coloring of G_1 or G_2, then $c(v) < c(w) < c(u)$. Therefore, G_1 and G_2 have no intercompatible colorings and the number of k-colorings of G, each counted with multiplicity equal to the number of intercompatible acyclic orientations of G is 0.

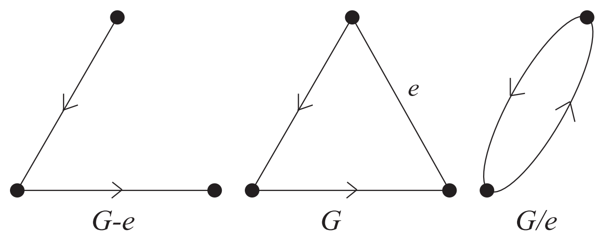

FIGURE 2. G and its contraction and deletion.

We now use contraction and deletion, with $e = uv$, to compute the chromatic polynomial of G. G/e (shown in Figure 2 with $G - e$) is a strongly connected directed graph, so $\chi_{G/e}(k) = k$. In $G - e$, there are $(k - i + 1)i$ proper k-colorings with $c(w) = i$. Therefore,

$$\chi_{G-e}(k) = \sum_{i=1}^{k} (k - i + 1)i = \frac{(k + 2)(k + 1)k}{3}$$

and so $\chi_G(k) = \frac{1}{3}(k + 2)(k + 1)k - k$. We can now see that Theorem 2 does not hold for G since $\chi_G(-2) = -2$.

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Department of Mathematics, San Francisco State University, San Francisco, CA 94132, USA
E-mail address: mattbeck@sfsu.edu

Department of Computing and Mathematical Sciences, California Institute of Technology, Pasadena CA 91125, USA
E-mail address: blado@caltech.edu

Department of Mathematics, Morehouse College, Atlanta GA 30314, USA
E-mail address: josephdcrawford@yahoo.com

Department of Mathematics, Amherst College, Amherst MA 01002, USA
E-mail address: tjeanlouis13@amherst.edu

Department of Mathematics, Iowa State University, Ames IA 50011, USA
E-mail address: myoung@iastate.edu