NOWHERE-ZERO $\vec{k}$-FLOWS ON GRAPHS

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Abstract. We introduce and study a multivariate function that counts nowhere-zero flows on a graph G, in which each edge of G has an individual capacity. We prove that the associated counting function is a piecewise-defined polynomial in these capacities, which satisfy a combinatorial reciprocity law that incorporates totally cyclic orientations of G.

Let $G = (V, E)$ be a graph, which we allow to have multiple edges. Fix an orientation of G, i.e., for each edge $e = uv \in E$ we assign one of u and v to be the head $h(e)$ and the other to be the tail $t(e)$ of e. A *flow* on (this orientation of) G is a labeling $\mathbf{x} \in A^E$ of the edges of G with values in an Abelian group A such that for every node $v \in V$,

$$\sum_{h(e)=v} x_e \ = \ \sum_{t(e)=v} x_e \,,$$

i.e., we have conservation of flow at each node. We are interested in counting such flows that are *nowhere zero*, i.e., $x_e \neq 0$ for every edge $e \in E$.

The case $A = \mathbb{Z}_k$ goes back to Tutte [11], who proved that the number $\overline{\varphi}(k)$ of nowhere-zero $\mathbb{Z}_k$-flows is a polynomial in k. (Tutte introduced this counting function as a dual concept to the chromatic polynomial.) In the case $A = \mathbb{Z}$, a *k-flow* takes on integer values in $\{-k+1, \dots, k-1\}$. The fact that the number $\varphi(k)$ of nowhere-zero k-flows is also a polynomial in k is due to Kochol [7]. Both $\overline{\varphi}(k)$ and $\varphi(k)$ are easily seen to be independent of the chosen orientation of G, and so we will write these flow polynomials as $\overline{\varphi}_G(k)$ and $\overline{\varphi}_G(k)$. We give three examples of each in (1) below. We also note the well-known fact that G admits no nowhere-zero flow if G has a *bridge*, i.e., an edge whose removal increases the number of components of G.

Since both flow counting functions are polynomials, it is natural to ask about evaluations other than at positive integers. Mirroring a famous result of Stanley [9] on chromatic polynomials and acyclic orientations of a graph, this question was answered for negative integers (instances of a *combinatorial reciprocity theorem*) by Beck–Zaslavsky [3] for $\varphi_G(k)$ and by Breuer–Sanyal [4] for $\overline{\varphi}_G(k)$; in both cases the counting function gives a connection to totally cyclic orientations of the graph.

Our goal is to introduce and study a flow counting function depending on several variables. We define a $\mathbf{k}$-*flow* on G, for $\mathbf{k} \in \mathbb{Z}_{>0}^E$, as an integral flow $\mathbf{x} \in \mathbb{Z}^E$ such that $|x_e| < k_e$ for each $e \in E$. In plain English, we have a different capacity for each edge. While this seems to be a natural concept for graph flows, we are not aware of any enumeration concept associated with it.

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As before, we count only the nowhere-zero $\mathbf{k}$-flows of G, and we denote this count (with a slight abuse of notation) by $\varphi_G(\mathbf{k})$. Below are three examples for this multivariate flow function.

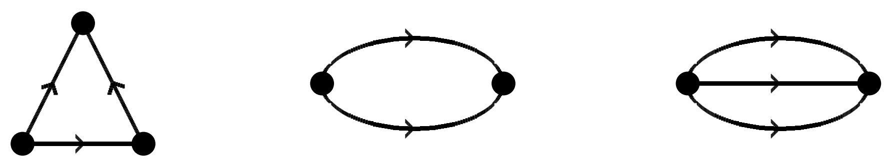

FIGURE 1. Three small graphs: K_3, $2K_2$, and $3K_2$.

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\begin{aligned}
\overline{\varphi}_{K_3}(k) &= k-1 \\
\varphi_{K_3}(k) &= 2(k-1) \\
\varphi_{K_3}(k_1,k_2,k_3) &= 2(k_1-1) \quad \text{if } k_1 \le k_2 \le k_3 \\
\overline{\varphi}_{2K_2}(k) &= k-1 \\
\varphi_{2K_2}(k) &= 2(k-1) \\
\varphi_{2K_2}(k_1,k_2) &= 2(k_1-1) \quad \text{if } k_1 \le k_2 \\
\overline{\varphi}_{3K_2}(k) &= (k-1)(k-2) \\
\varphi_{3K_2}(k) &= 3(k-1)(k-2) \\
\varphi_{3K_2}(k_1,k_2,k_3) &= \begin{cases} (2k_1-2)(2k_2-3) & \text{if } k_1 \le k_2 \le k_3 > k_2+k_1\,, \\ -k_1^2+2k_1k_2+2k_1k_3-5k_1-k_2^2 & \\ \qquad +2k_2k_3-3k_2-k_3^2-k_3+6 & \text{if } k_1 \le k_2 \le k_3 \le k_2+k_1\,. \end{cases}
\end{aligned} \tag{1}
$$

Already in these small examples, we can see some similarities (e.g., the degrees of the flow polynomials associated with a given graph, or the constant terms of $\varphi_G(k)$ and $\varphi_G(\mathbf{k})$) and some differences among the three flow counting functions, most notably the fact that $\varphi_G(\mathbf{k})$ is only a *piecewise-defined* polynomial in $\mathbf{k}$ (note that in (1) we state $\varphi_G(\mathbf{k})$ in each case only for $k_1 \le k_2 \le \cdots$; we may do so in these cases due to the symmetry of the graphs considered). A moment's thought reveals that this is structurally the best we can hope for $\varphi_G(\mathbf{k})$, and in fact, our first result says this structure always holds. For a graph $G = (V, E)$, let $\xi_G := |E| - |V| + \#\text{components}(G)$, the *cyclomatic number* of G.

Theorem 1. *For a bridgeless graph G, the multivariate flow counting function $\varphi_G(\mathbf{k})$ is a piecewise-defined polynomial of degree ξ_G.*

Our second result is a combinatorial reciprocity theorem for $\varphi_G(\mathbf{k})$, which gives a vector-valued generalization of the main result in [3]. To state it, recall that a *totally cyclic orientation* of G is an orientation for which ever edge lies in a coherently-oriented cycle. A $\mathbf{k}$-flow of G gives naturally rise to orientations of G by switching the directions of those edges of the initial orientation of G that have a negative flow label and possibly switching some of the edges with a zero flow. Any such orientation is *compatible* with the $\mathbf{k}$-flow. Denote by $\mathbf{1} \in \mathbb{Z}^E$ a vector all of whose entries are 1.

Theorem 2. *For a bridgeless graph G, $(-1)^{\xi_G}\varphi_G(-\mathbf{k})$ equals the number of $(\mathbf{k}+\mathbf{1})$-flows of G, each counted with multiplicity equal to the number of compatible totally cyclic orientations of G. In particular, $(-1)^{\xi_G}\varphi_G(\mathbf{0})$ equals the number of totally cyclic orientations of G.*

Proof of Theorem 1. Given a graph G with a fixed orientation, let $A \in \mathbb{Z}^{V \times E}$ be its *(signed) incidence matrix* with entries

$$a_{ve} = \begin{cases} 1 & \text{if } v = h(e), \\ -1 & \text{if } v = t(e), \\ 0 & \text{otherwise.} \end{cases}$$

Let F be the kernel of A, viewed as a subspace of $\mathbb{R}^E$. Note that, by definition, a nowhere-zero integral flow of G is an integer lattice point in

$$F \setminus \mathcal{H} \qquad \text{where} \qquad \mathcal{H} := \{x_e = 0 : \, e \in E\} \, ,$$

the arrangement of coordinate hyperplanes in $\mathbb{R}^E$. The induced arrangement in F has regions (i.e., maximal connected components of $F \setminus \mathcal{H}$) which are in one-to-one correspondence with the totally cyclic orientations of G [6]. Taking a leaf from the geometric setup of [2] (which was also used in [3]), given $\mathbf{k} \in \mathbb{Z}^E_{>0}$, a $\mathbf{k}$-flow is precisely a point in

$$\left([-k_1, k_1] \times \cdots \times [-k_{|E|}, k_{|E|}] \right)^\circ \cap \mathbb{Z}^E \cap F \setminus \mathcal{H} \, .$$

(Here $\mathcal{P}^\circ$ denotes the interior of $\mathcal{P}$.) This geometric object is a union of open polytopes $\mathcal{P}^\circ_\sigma$ (depending on $\mathbf{k}$) which are naturally indexed by the totally cyclic orientations of G. More precisely, if we let

$$\mathcal{P}_\sigma(\mathbf{k}) := \left([0, k_1] \times \cdots \times [0, k_{|E|}] \right) \cap F_\sigma \, ,$$

where F_σ denotes the kernel of the incidence matrix A_σ of G reoriented by σ,

$$f_\sigma(\mathbf{k}) := \# \left(\mathcal{P}_\sigma(\mathbf{k})^\circ \cap \mathbb{Z}^E \right) ,$$

and Ω the set of all the totally cyclic orientations, then

$$\varphi_G(\mathbf{k}) = \sum_{\sigma \in \Omega} f_\sigma(\mathbf{k}). \tag{2}$$

Thus it suffices to prove that $\# \left(\mathcal{P}_\sigma(\mathbf{k})^\circ \cap \mathbb{Z}^E \right)$ is a piecewise-defined polynomial of degree ξ_G. The structure of this counting function is that of a *vector partition function*:

$$\# \left(\mathcal{P}_\sigma(\mathbf{k})^\circ \cap \mathbb{Z}^E \right) = \# \left\{ \mathbf{x} \in \mathbb{Z}^E : \, A_\sigma \, \mathbf{x} = 0 \, , \; 0 < x_e < k_e \right\} .$$

It is well known (see, e.g., [8]) that A_σ is *totally unimodular*, i.e., every square submatrix of A_σ has determinant ± 1 or 0. But from this we can conclude with the general theory of vector partition functions (see, e.g., [5, 10]) that $\# \left(\mathcal{P}_\sigma(\mathbf{k})^\circ \cap \mathbb{Z}^E \right)$ is a piecewise-defined polynomial. Its degree equals ξ_G because that is the dimension of the underlying polytope. □

Proof of Theorem 2. We start with (2) and use [1]:

$$\varphi_G(-\mathbf{k}) = \sum_{\sigma \in \Omega} f_\sigma(-\mathbf{k}) = \sum_{\sigma \in \Omega} (-1)^{\xi_G} \, \# \left(\mathcal{P}_\sigma(\mathbf{k}) \cap \mathbb{Z}^E \right) .$$

What we are counting on the right-hand side (ignoring $(-1)^{\xi_G}$) are $(\mathbf{k}+\mathbf{1})$-flows with multiplicities that come from zero entries. This multiplicity is precisely the number of closed regions of the hyperplane arrangement induced by $\mathcal{H}$ on F (using the notation from the beginning of our proof of Theorem 1) a $(\mathbf{k}+\mathbf{1})$-flow lies in (viewed as a point in F). The afore-mentioned interpretation of these regions in terms of totally cyclic orientations of G [6] gives Theorem 2. □

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