THE MAXIMUM NULLITY OF A COMPLETE SUBDIVISION GRAPH IS EQUAL TO ITS ZERO FORCING NUMBER, $\mathrm{M}(\mathring{\widetilde{G}}) = \mathrm{Z}(\mathring{\widetilde{G}})$ [*]

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Abstract. Barrett et al. asked in "Minimum rank of edge subdivisions of graphs," *Electronic Journal of Linear Algebra* (2009) 18: 530–563, whether the maximum nullity is equal to the zero forcing number for all complete subdivision graphs. We prove that this equality holds and determine this common value.

Key words. zero forcing number, maximum nullity, minimum rank, complete subdivision, matrix, graph

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1. Introduction.

A *complete subdivision graph* $\mathring{\widetilde{G}}$ is obtained from a graph G by subdividing every edge of G once; $\mathrm{M}(F, G)$ denotes the maximum nullity of G over a field F and $\mathrm{Z}(G)$ denotes the zero forcing number of G (definitions are given below). The paper [4] concludes with a list of open questions, including the following fundamental problem.

QUESTION 1.1. *Is* $\mathrm{M}(F, \mathring{\widetilde{G}}) = \mathrm{Z}(\mathring{\widetilde{G}})$ *for every field F and graph G?*

The main result of this paper is an affirmative answer to this question (see Theorem 1.3 below), which is proved by giving a formula for this common value and

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giving constructive methods for producing a zero forcing set of the required cardinality (Section 2) and a matrix in $\mathcal{S}(F,\widetilde{G})$ of the required nullity (Section 3). The proofs also produce information about minimum zero forcing sets and optimal matrices of complete subdivision graphs.

We now define our terminology, including terms basic to the problem and new terms needed to state the common bound. For a (simple, undirected) graph G, $n(G)$ denotes the number of vertices (order) of G and $m(G)$ denotes the number of edges (size) of G (we can use m and n when G is clear from context). Let F be any field. For a graph G that has vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$, $\mathcal{S}(F,G)$ is the set of all symmetric $n \times n$ matrices A with entries from F such that for any $i \neq j$, $a_{ij} \neq 0$ if and only if $\{i,j\} \in E(G)$. The *minimum rank* of G is $\operatorname{mr}(F,G) = \min\{\operatorname{rank} A : A \in \mathcal{S}(F,G)\}$, and the *maximum nullity* of G is $\operatorname{M}(F,G) = \max\{\operatorname{null} A : A \in \mathcal{S}(F,G)\}$. If the field F is omitted, it is assumed to be the real numbers: $\operatorname{mr}(G) = \operatorname{mr}(\mathbb{R},G)$ and $\operatorname{M}(G) = \operatorname{M}(\mathbb{R},G)$. Note that for any field F, $\operatorname{mr}(F,G) + \operatorname{M}(F,G) = n(G)$, so the problem of determining the minimum rank of a given graph is equivalent to the problem of determining its maximum nullity.

The zero forcing number of a graph is the minimum number of blue vertices initially needed to color all vertices blue according to the *color-change rule*, defined as follows: If G is a graph with each vertex colored either white or blue, b is a blue vertex of G and exactly one neighbor w of b is white, then change the color of w to blue. In this case we say b *forces* w and write $b \to w$. Let S be a subset of V. The *final coloring of* S is the result of initially coloring every vertex in S blue and every vertex in $V(G) \setminus S$ white, and then applying the color-change rule until no more changes are possible. A *zero forcing set* of G is a set $Z \subseteq V(G)$ such that every vertex in the final coloring of Z is blue. The *zero forcing number* of G is $\operatorname{Z}(G) = \min\{|Z| : Z \text{ is a zero forcing set of } G\}$ and $\operatorname{mz}(G) = n(G) - \operatorname{Z}(G)$. A zero forcing set Z is called a *minimum zero forcing set* of G if $|Z| = \operatorname{Z}(G)$. The terminology 'zero forcing' refers to the fact that using zero forcing on $\mathcal{G}(A)$ corresponds to forcing certain entries in a null vector of A to be zero, and it was established in [1] that for any field F and graph G, $M(F,G) \leq \operatorname{Z}(G)$, or equivalently, $\operatorname{mz}(G) \leq \operatorname{mr}(F,G)$. Given a zero forcing set Z of G, a *zero forcing process* for Z is some set of forces that can be used to color all the vertices blue. The forces in a zero forcing process can be grouped into induced paths, called *forcing paths*, each beginning with a vertex in Z. Note that the forcing paths are not uniquely determined by Z. A vertex w is Z*-terminal* (for a particular zero forcing process of Z) if w is the last vertex in a zero forcing path of the zero forcing process (it is possible that $v \in Z$ is also Z-terminal, if the path is a single vertex).

The vertices of the complete subdivision $\widetilde{G}$ of G are of two types: the *original vertices* $V(G)$ and the *edge-vertices*, which are the new vertices created by edge subdivision. Each edge-vertex of $\overset{\circ}{G}$ corresponds to an edge of G, and we sometimes use the same symbol for both the edge of G and the edge-vertex of $\overset{\circ}{G}$. A *bridge* or *cut-edge* of a connected graph is an edge whose deletion disconnects the graph. A *bridgeless* graph is a connected graph with no bridge; necessarily such a graph does not have order 2. A 2-*edge connected* graph is a connected graph from which at least two edges must be deleted to disconnect it. A single vertex is bridgeless but not 2-edge connected. A graph is *minimally* 2-*edge connected* if it is 2-edge connected and the deletion of any edge leaves a graph that is not 2-edge connected, i.e., has a bridge. An *island* of a connected graph is a maximal bridgeless subgraph, necessarily induced. A *cut-vertex* of a connected graph is a vertex whose deletion disconnects the graph.

Definition 1.2. Given a graph G, define the *bridge forest* of G to be the forest $BF(G)$ obtained by contracting every island with more than one vertex to a single vertex. When G is connected the bridge forest is a tree, and we often refer to it as the *bridge tree*.

The zero forcing number of a forest, and hence of a subdivision of a forest, is readily computed by a variety of algorithms (e.g., see [7]). Our main result is the following:

Theorem 1.3. *For any graph G with c connected components and any field F,*

$$\mathrm{M}(F,\overset{\circ}{G}) = \mathrm{Z}(\overset{\circ}{G}) = m(G) - n(G) + c + \mathrm{Z}(\overset{\circ}{\overline{BF(G)}}).$$

The result is proved in the case that G is connected by giving constructions of a zero forcing set of cardinality $m(G)-n(G)+1+\mathrm{Z}(\overset{\circ}{\overline{BF(G)}})$ (Section 2) and a matrix in $\mathcal{S}(\overset{\circ}{G})$ of nullity $m(G)-n(G)+1+\mathrm{Z}(\overset{\circ}{\overline{BF(G)}})$ (Section 3). Additivity of the parameters used completes the proof for all graphs.

The equality $\mathrm{M}(F,\overset{\circ}{G}) = \mathrm{Z}(\overset{\circ}{G})$ has already been established for graphs that have a Hamilton path [4] or that do not have a bridge [5]. We will use results from these papers.

Theorem 1.4. [4, Corollary 3.13] *For any connected graph G and field F, $\mathrm{mr}(F,\overset{\circ}{G}) \leq 2n(G) - 2$, or equivalently, $\mathrm{M}(F,\overset{\circ}{G}) \geq m(G) - n(G) + 2$.*

Let $\mathcal{K}$ be the family of bipartite graphs $G = (V(G), E(G))$ such that there is a bipartition of the vertices $V(G) = X \,\dot\cup\, Y$ with $\deg x \leq 2$ for all $x \in X$ [5]. Clearly every complete edge subdivision graph is in $\mathcal{K}$. A graph $G \in \mathcal{K}$ is *special* if for every field F there exists a matrix $A \in \mathcal{S}(F,G)$ such that:

1. $\mathrm{null}\, A = \mathrm{M}(F,G)$, and
2. if $x \in X(G)$, then $a_{xx} = 0$.

Theorem 1.5. [5, Theorem 2.16] *If G is a graph in $\mathcal{K}$ that does not have a bridge, then G is special and* $\mathrm{M}(F,G) = \mathrm{Z}(G)$ *for every field F.*

Note that a matrix $A \in \mathbb{Z}^{n\times n} \subset \mathbb{Q}^{n\times n} \subset \mathbb{R}^{n\times n}$ can also be interpreted as living in ${\mathbb{Z}_p}^{n\times n}$ for any prime p, and we denote the graph when viewing A this way by $\mathcal{G}^{\mathbb{Z}_p}(A)$ (for F a field of characteristic p, $\mathcal{G}^F(A) = \mathcal{G}^{\mathbb{Z}_p}(A)$). A symmetric integer matrix A has $\mathcal{G}^F(A) = \mathcal{G}(A)$ for all fields F if and only if all off-diagonal entries of A are in $\{0, \pm 1\}$. A *universally optimal matrix* is an integer matrix A such that every off-diagonal entry of A is 0, 1, or -1, and for all fields F, $\operatorname{rank}^F(A) = \operatorname{mr}(F, \mathcal{G}(A))$.

Remark 1.6. The following technique was used extensively in [6]: If A is a symmetric integer matrix with all off-diagonal entries in $\{0, \pm 1\}$ with $\operatorname{rank}^{\mathbb{R}} A = \operatorname{mz}(\mathcal{G}(A))$, then $\mathcal{G}(A)$ has field independent minimum rank and A is a universally optimal matrix for $\mathcal{G}(A)$ because $\operatorname{mz}(\mathcal{G}(A)) \le \operatorname{rank}^F A \le \operatorname{rank}^{\mathbb{R}} A = \operatorname{mz}(\mathcal{G}(A))$.

2. Bounding zero forcing number from above. In this section we establish the upper bound for $\mathrm{Z}(\vec{G})$ by producing a zero forcing set. For a graph G, an *orientation* $\vec{G}$ of G is obtained by assigning a direction to each edge. The *oriented vertex-edge incidence matrix* of $\vec{G}$ is the matrix $Q = [q_{ve}]$ where for directed edge $e = (u, v)$, $q_{ue} = -1$, $q_{ve} = 1$, and $q_{we} = 0$ for $w \ne u, v$.

Lemma 2.1. *If Y is the vertex-edge incidence pattern of a connected graph on n vertices, then* $\operatorname{mr}(Y) = n - 1$*, and we can order the vertices and edges so that Y has the form*

$$\begin{bmatrix} \times & \otimes & \otimes & \cdots & \otimes & \cdots & \otimes \\ \times & \otimes & \otimes & \cdots & \otimes & \cdots & \otimes \\ 0 & \times & \otimes & \cdots & \otimes & \cdots & \otimes \\ \vdots & \ddots & \ddots & \ddots & \vdots & \cdots & \otimes \\ 0 & \cdots & 0 & \times & \otimes & \cdots & \otimes \end{bmatrix} \tag{2.1}$$

where $\times$ denotes a nonzero entry, and $\otimes$ denotes an entry that may be zero or nonzero.

Proof. Any orientation of the graph produces an oriented vertex-edge incidence matrix having Y as its pattern and having the all ones vector as a left null vector, so $\operatorname{mr}(Y) \le n - 1$. To achieve the form (2.1), number the vertices and edges of a spanning tree as follows: v_1 is any vertex; given $v_1, \dots, v_k$, e_k is any edge incident with exactly one of $v_1, \dots, v_k$ and v_{k+1} is the other endpoint of e_k. Finally number the rest of the edges in any order. Any pattern of the form (2.1) has minimum rank at least $n-1$ by considering the subpattern in rows 2 through n and columns 1 through $n - 1$. □

Proposition 2.2. *If G is connected and $\mathrm{M}(\overleftrightarrow{G}) = m(G) - n(G) + 2 = \mathrm{Z}(\overleftrightarrow{G})$, then the minimum rank of $\overleftrightarrow{G}$ is field independent and $\overleftrightarrow{G}$ has a universally optimal matrix.*

Proof. Observe that $\mathrm{M}(\overleftrightarrow{G}) = m(G) - n(G) + 2$ is equivalent to $\mathrm{mr}(\overleftrightarrow{G}) = 2n(G) - 2$. Let Q be an oriented vertex-edge incidence matrix of $\vec{G}$ (for some orientation of G), so $\operatorname{rank} Q = n(G) - 1$. Then for $A = \begin{bmatrix} 0 & Q \\ Q^{\mathrm{T}} & 0 \end{bmatrix}$, $A \in \mathcal{S}(\overleftrightarrow{G})$, $\operatorname{rank} A = 2(n(G) - 1) = \mathrm{mr}(\overleftrightarrow{G})$, and by Remark 1.6, A is a universally optimal matrix. □

Theorem 2.3. *Suppose G is connected and there exists a real matrix $A \in \mathcal{S}(\overleftrightarrow{G})$ such that $\operatorname{rank} A = \mathrm{mr}(\overleftrightarrow{G})$ and all diagonal entries of A associated with edge-vertices of $\overleftrightarrow{G}$ are zero. Then $\mathrm{mr}(\overleftrightarrow{G}) = 2n(G) - 2$ and $\mathrm{M}(\overleftrightarrow{G}) = m(G) - n(G) + 2$.*

Proof. Let $n = n(G)$. Number the vertices of $\overleftrightarrow{G}$ with original vertices first and edge-vertices second and numbered so as to achieve the pattern given in (2.1). Then the matrix A has the form $A = \begin{bmatrix} D & B \\ B^{\mathrm{T}} & 0 \end{bmatrix}$ where B has the pattern given in (2.1) and D is a diagonal matrix. Delete the first row and first column of A to obtain $A(1) = \begin{bmatrix} D(1) & T & C \\ T^{\mathrm{T}} & 0 & 0 \\ C^{\mathrm{T}} & 0 & 0 \end{bmatrix}$ where T is an $(n-1) \times (n-1)$ upper triangular matrix with nonzero diagonal. Since $\det \begin{bmatrix} D(1) & T \\ T^{\mathrm{T}} & 0 \end{bmatrix} = \pm(\det T)^2 \neq 0$, $\operatorname{rank} A \geq 2(n-1)$. Since $\operatorname{rank} A = \mathrm{mr}(\overleftrightarrow{G})$, $\mathrm{mr}(\overleftrightarrow{G}) \geq 2n - 2$; equality follows by Theorem 1.4. □

The next corollary is immediate from Theorems 1.5 and 2.3, and Proposition 2.2.

Corollary 2.4. *If G is connected and does not have a bridge then for every field F,*

$$\mathrm{Z}(\overleftrightarrow{G}) = \mathrm{M}(F, \overleftrightarrow{G}) = m(G) - n(G) + 2, \qquad \mathrm{mz}(\overleftrightarrow{G}) = \mathrm{mr}(F, \overleftrightarrow{G}) = 2n(G) - 2,$$

and G has a universally optimal matrix.

Theorem 2.5. *Given any connected bridgeless graph G, and any vertices u, v of G (not necessarily distinct), there exists a zero forcing set Z of $\overleftrightarrow{G}$ of order $m(G) - n(G) + 2$ (necessarily minimum) such that $u \in Z$ and v is Z-terminal.*

Proof. The proof is by induction on the number of vertices $n(G)$. The result is clear for a single vertex. Assume that for any connected bridgeless graph G' with $n(G') < n$ and any vertices u, v of G', there exists a zero forcing set Z of $\overleftrightarrow{G'}$ of order $m(G') - n(G') + 2$ such that $u \in Z$ and v is Z-terminal.

Let G be a connected bridgeless graph with $n(G) = n > 1$ (so G is 2-edge connected). Remove edges $f_1, \ldots, f_\ell$ from G to obtain a minimally 2-edge connected graph H; note that $n(H) = n(G)$ and $m(H) = m(G) - \ell$. Choose any edge e of H. Then $H - e$ necessarily has a bridge (or H would not have been *minimally* 2-edge connected). The bridge forest of $H - e$ is necessarily a path (or H would not have been 2-edge connected). The graph H consists of the $k \geq 2$ islands of $H - e$, connected cyclically with a single edge between each consecutive pair in the cycle (see Figure 2.1).

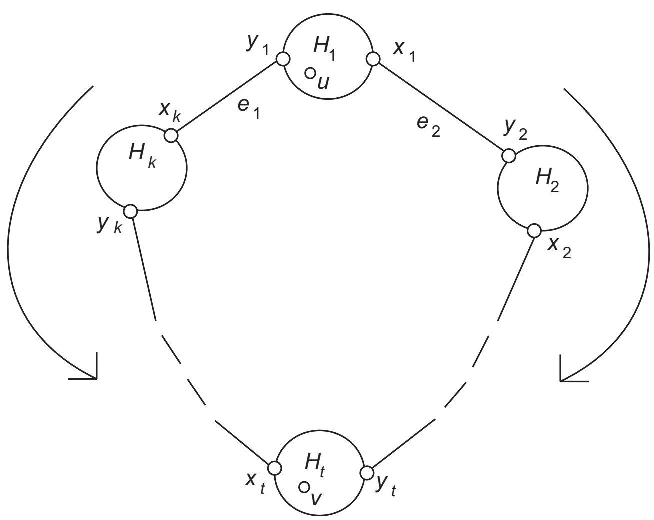

FIG. 2.1. *Notation for Theorem 2.5; arrows indicate the direction of zero forcing.*

Since we are working over a cycle of order k, subscript arithmetic will be taken modulo k. Let H_1 be the island of $BF(H-e)$ containing u, and number the remaining islands of $BF(H)$ as $H_2, \ldots, H_k$ in cycle order. Number the edges having endpoints in two different islands in cycle order as $e_i = \{x_{i-1}, y_i\}$ with $x_i, y_i \in V(H_i)$ (it is possible $x_i = y_i$). Let t denote the index of the island containing vertex v (the argument below assumes $t \neq 1$ but a minor modification handles the case $t = 1$). The notation used is illustrated in Figure 2.1.

Since $n(H_i) < n(H) = n$ for $i = 1, \ldots, k$, the induction hypothesis applies to the islands H_i. We wish to construct a zero forcing set for $\overset{\circ}{\overline{H}}$ of cardinality $m(H) - n(H) + 2$, using certain zero forcing sets for the subdivided islands $\overset{\circ}{\overline{H_i}}$. For $1 < i < t$, choose a zero forcing set Z_i for $\overset{\circ}{\overline{H_i}}$ with $y_i \in Z_i$ and x_i being Z_i-terminal. For $t < i \leq k$, choose a zero forcing set Z_i for $\overset{\circ}{\overline{H_i}}$ with $x_i \in Z_i$ and y_i being Z_i-terminal. For $\overset{\circ}{\overline{H_t}}$ choose a minimum zero forcing set Z_t with $y_t \in Z_t$ and v being Z_t-terminal. For $\overset{\circ}{\overline{H_1}}$ choose a minimum zero forcing set Z_1 with $u \in Z_1$ and x_1 being Z_1-terminal.

Define

$$Z := \bigcup_{i=2}^{t-1}(Z_i \setminus \{y_i\}) \cup \bigcup_{i=t+1}^{k}(Z_i \setminus \{x_i\}) \cup (Z_t \setminus \{y_t\}) \cup Z_1 \cup \{e_1\}.$$

Observe that $|Z| = \sum_{i=2}^{k}(|Z_i| - 1) + |Z_1| + 1$. By the induction hypothesis, $|Z_i| = m(H_i) - n(H_i) + 2$, and therefore

$$|Z| = \sum_{i=1}^{k}(m(H_i) - n(H_i) + 1) + 2 = \sum_{i=1}^{k} m(H_i) + k - \sum_{i=1}^{k} n(H_i) + 2 = m(H) - n(H) + 2.$$

Start the zero forcing process that produces x_1 as Z_1-terminal on $\mathring{\overline{H_1}}$. Because $e_1 \in Z$, the zero forcing process within $\mathring{\overline{H_1}}$ runs to completion. For $i < t$, when the zero forcing process on $\mathring{\overline{H_{i-1}}}$ is complete (so x_{i-1} is blue), force the vertices e_i and y_i. Then completely perform forcing on $\mathring{\overline{H_i}}$ to obtain that x_i is Z_i-terminal (in $\mathring{\overline{H_i}}$). For $i > t$, when the zero forcing process on $\mathring{\overline{H_{i+1}}}$ is complete (so y_{i+1} is blue), force the vertices e_{i+1} and x_i. Then perform forcing on $\mathring{\overline{H_i}}$ to obtain that y_i is Z_i-terminal (in $\mathring{\overline{H_i}}$). Finally, $y_{t+1} \to e_{t+1}$ and $x_{t-1} \to e_t \to y_t$, and perform forcing in $\mathring{\overline{H_t}}$ to obtain that v is Z_t-terminal in $\mathring{\overline{H_t}}$ and hence in $\mathring{\overline{H}}$.

Finally, let $\widehat{Z}$ be the union of Z and the set of the edge-vertices $f_1, \ldots, f_\ell$ of $\mathring{\overline{G}}$ associated with the deleted edges of G. Then $\widehat{Z}$ is a zero forcing set for $\mathring{\overline{G}}$, $|\widehat{Z}| = |Z| + \ell = m(G) - n(G) + 2$, $u \in \widehat{Z}$ and v is $\widehat{Z}$-terminal (using the same zero forcing process as in $\mathring{\overline{H}}$). □

Theorem 2.6. *For any connected graph G,*

$$\mathrm{Z}(\mathring{\overline{G}}) \le m(G) - n(G) + 1 + \mathrm{Z}(\mathring{\overline{BF(G)}}).$$

Proof. Construct the bridge tree of G and subdivide it to obtain $\mathring{\overline{BF(G)}}$. Choose a zero forcing set $B = \{b_1, \ldots, b_z\}$ for $\mathring{\overline{BF(G)}}$ (where $z = \mathrm{Z}(\mathring{\overline{BF(G)}})$) and choose a set of forcing paths $P^{(i)}$ with $b_i \in V(P^{(i)})$. Number the vertices in $\mathring{\overline{BF(G)}}$ so that the jth vertex in path $P^{(i)}$ (in forcing order) is numbered $w_j^{(i)}$ (so $b_i = w_1^{(i)}$). The islands of G and edge-vertices of $\mathring{\overline{BF(G)}}$ will collectively be named $H_j^{(i)}$ in such a way that $\mathring{\overline{H_j^{(i)}}}$ is always the island corresponding to vertex $w_j^{(i)}$ of the tree $\mathring{\overline{BF(G)}}$. Depending on j, $H_j^{(i)}$ is an island vertex of G, a multiple-vertex island of G, or a single edge-vertex of $\mathring{\overline{BF(G)}}$.

Within $\mathring{\overline{H_j^{(i)}}}$, let $x_j^{(i)}$ be the vertex that is the endpoint of the bridge from $\mathring{\overline{H_j^{(i)}}}$ to $\mathring{\overline{H_{j+1}^{(i)}}}$ (if there is such), and let $y_j^{(i)}$ be the vertex that is the endpoint of the bridge from $\overline{H_j^{(i)}}$ to $\overline{H_{j-1}^{(i)}}$ (if there is such); it is possible $x_j^{(i)} = y_j^{(i)}$. Figure 2.2 illustrates this nomenclature.

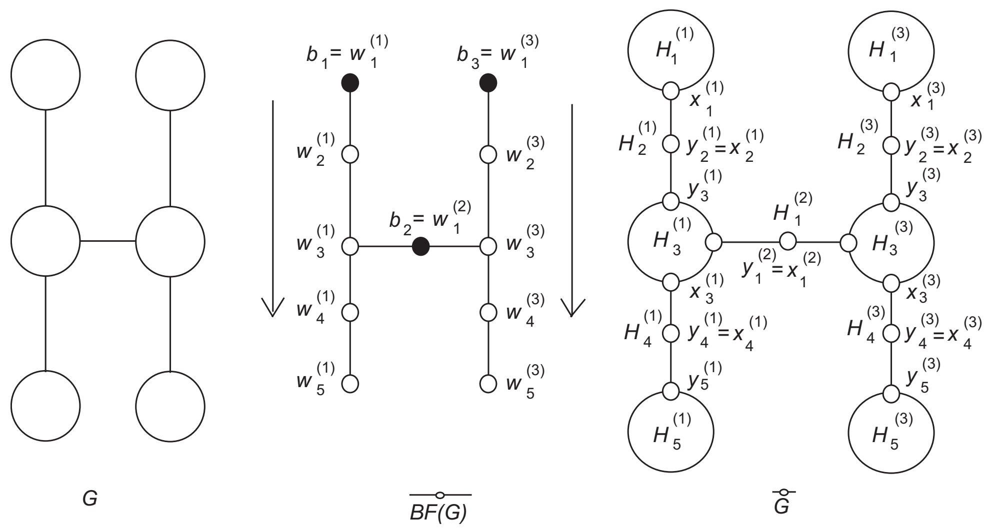

FIG. 2.2. *Example for Theorem 2.6; arrows indicate the direction of zero forcing.*

Then construct a zero forcing set as follows: For each island $\overline{H_j^{(i)}}$ corresponding to an original island $H_j^{(i)}$ of G, choose a zero forcing set $Z_j^{(i)}$ of order $m(H_j^{(i)}) - n(H_j^{(i)}) + 2$ with $y_j^{(i)} \in Z_j^{(i)}$ and $x_j^{(i)}$ being $Z_j^{(i)}$-terminal (if one or the other of $x_j^{(i)}, y_j^{(i)}$ does not exist, ignore that instruction). For an edge-vertex island, the zero forcing set is the single vertex $x_j^{(i)} = y_j^{(i)}$. Then for all i, j define

$$\widehat{Z}_j^{(i)} := \begin{cases} Z_j^{(i)} & \text{if } j = 1; \\ Z_j^{(i)} \setminus \{y_j^{(i)}\} & \text{if } j > 1. \end{cases}$$

Then

$$Z := \bigcup_{i,j} \widehat{Z}_j^{(i)}$$

is a zero forcing set with the following zero forcing process: For each i, force in $\overline{H_1^{(i)}}$ with x_1 being $Z_1^{(i)}$-terminal. Then proceed through the paths as the forcing is done in the tree, with $x_j^{(i)} \to y_{j+1}^{(i)}$.

Let h be the number of islands of G (so $BF(G)$ has $h-1$ edges). Observe that

$$|Z| = \sum_{i,j} |\widehat{Z}_j^{(i)}| = \sum_{i,j} (|Z_j^{(i)}| - 1) + \mathrm{Z}(\overline{BF(G)}).$$

If $\overset{\circ}{\overline{H_j^{(i)}}}$ is an edge-vertex of $\overset{\circ\circ}{G}$ then $\widehat{Z}_j^{(i)} = \emptyset$, or equivalently, $|Z_j^{(i)}| - 1 = 0$. So the sum can be taken only over the subdivisions $\overset{\circ}{\overline{H_j^{(i)}}}$ of the islands $H_j^{(i)}$ of G, and for each such subdivided island, $|Z_j^{(i)}| = m(H_j^{(i)}) - n(H_j^{(i)}) + 2$. Since $n(G) = \sum_{\text{islands of } G} n(H_j^{(i)})$ and $m(G) = \left(\sum_{\text{islands of } G} m(H_j^{(i)}) \right) + h - 1$,

$$\begin{aligned} |Z| &= \sum_{\text{islands of } G} (m(H_j^{(i)}) - n(H_j^{(i)}) + 1) + \mathrm{Z}(\overset{\circ}{\overline{BF(G)}}) \\ &= \sum_{\text{islands of } G} m(H_j^{(i)}) - \sum_{\text{islands of } G} n(H_j^{(i)}) + h + \mathrm{Z}(\overset{\circ}{\overline{BF(G)}}) \\ &= m(G) - n(G) + 1 + \mathrm{Z}(\overset{\circ}{\overline{BF(G)}}). \end{aligned}$$

□

3. Bounding maximum nullity. In this section we determine $\mathrm{M}(\overset{\circ\circ}{G})$ by using techniques that produce a matrix of the desired nullity.

THEOREM 3.1. *Let G be a graph constructed by appending $\ell \geq 0$ leaves to an island H. Then for any field F,*

$$\mathrm{M}(F, \overset{\circ\circ}{G}) = \mathrm{Z}(\overset{\circ\circ}{G}) = m(G) - n(G) + 1 + \mathrm{Z}(\overset{\circ}{\overline{BF(G)}}). \tag{3.1}$$

If $\ell \geq 2$, this formula is equivalent to

$$\mathrm{M}(F, \overset{\circ\circ}{G}) = m(H) - n(H) + \ell \quad \text{or} \quad \mathrm{mr}(F, \overset{\circ\circ}{G}) = 2n(H) + \ell. \tag{3.2}$$

Finally, $\overset{\circ\circ}{G}$ has a universally optimal matrix and field independent minimum rank.

Proof. If $\ell = 0, 1$ or 2, then $\overset{\circ}{\overline{BF(G)}}$ is P_1, P_2 or P_3, so $\mathrm{Z}(\overset{\circ}{\overline{BF(G)}}) = 1$, and thus $m(G) - n(G) + 2 \leq \mathrm{M}(F, \overset{\circ\circ}{G}) \leq \mathrm{Z}(\overset{\circ\circ}{G}) \leq m(G) - n(G) + 2$, where the first inequality is by Theorem 1.4 and the last by Theorem 2.6. Furthermore, G has a universally optimal matrix and field independent minimum rank by Proposition 2.2.

Suppose $\ell \geq 2$. Then $BF(G) = K_{1,\ell}$, so $\mathrm{Z}(\overset{\circ}{\overline{BF(G)}}) = \ell - 1$. Since $m(G) - n(G) = m(H) - n(H)$, in this case the equivalence of (3.1) and (3.2) is clear. By Theorem 2.6 and Remark 1.6, it suffices to exhibit a $\{0, 1\}$ matrix $A \in \mathcal{S}(F, \overset{\circ\circ}{G})$ having $\operatorname{null} A \geq m(H) - n(H) + \ell$. Because $n(\overset{\circ\circ}{G}) = n(G) + m(G) = n(H) + m(H) + 2\ell$, $\operatorname{null} A \geq m(H) - n(H) + \ell$ is equivalent to $\operatorname{rank} A \leq 2n(H) + \ell$.

For each original vertex u of $\overset{\circ\circ}{H}$, let A_u be the adjacency matrix of rank 2 of the star formed by u and its neighbors in $\overset{\circ\circ}{G}$. Embed A_u appropriately into a matrix of order $n(\overset{\circ\circ}{G})$ to obtain a matrix $\widetilde{A}_u$ of rank 2. Similarly, for each leaf vertex v_i,

$i = 1, \ldots, \ell$, let J_{v_i} be the 2×2 matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ of rank 1 corresponding to v_i, its neighbor in $\mathring{\widetilde{G}}$, and their common edge. Embed J_{v_i} appropriately into a matrix of order $n(\mathring{\widetilde{G}})$ to obtain a matrix $\widetilde{J}_{v_i}$ of rank 1. Let

$$A = \sum_{u \in V(H)} \widetilde{A}_u + \sum_{i=1}^{\ell} \widetilde{J}_{v_i}.$$

Then A is a $\{0,1\}$ matrix in $\mathcal{S}(F, \mathring{\widetilde{G}})$ and has rank no more than $2n(H) + \ell$. □

Before giving the proof of our main result on nullity, we will need a basic formula to allow us to look at the nullity when splitting along an edge in a subdivided graph. In the following we will let $G \underset{e}{+} H$ denote the graph formed by taking the disjoint union of G and H and adding the edge $e = \{x, y\}$ which connects vertex $x \in G$ to $y \in H$. This graph was called an *edge sum* in [3] and the range of the minimum rank of the edge sum was determined. Similarly, identifying x and y to a common vertex v gives the graph we denote by $G \underset{v}{\oplus} H$, which has v as a cut-vertex.

LEMMA 3.2. *Let $G = G_1 \underset{e}{+} G_2$ be a graph with bridge $e = \{x, y\}$. Then*

$$\mathrm{M}(\mathring{\widetilde{G}}) = \mathrm{M}(\mathring{\overline{G_1}} \underset{x}{\oplus} K_2) + \mathrm{M}(\mathring{\overline{G_2}} \underset{y}{\oplus} K_2) - 1.$$

Proof. By the cut-vertex reduction formula (see, e.g., [7])

$$\mathrm{mr}(\mathring{\widetilde{G}}) = \min\left\{ \mathrm{mr}(\mathring{\overline{G_1}} \underset{x}{\oplus} K_2) + \mathrm{mr}(\mathring{\overline{G_2}} \underset{y}{\oplus} K_2), \mathrm{mr}(\mathring{\overline{G_1}}) + \mathrm{mr}(\mathring{\overline{G_2}}) + 2 \right\}.$$

But for any graph H, we have $\mathrm{mr}(H \oplus K_2) \le \mathrm{mr}(H) + 1$, so

$$\mathrm{mr}(\mathring{\widetilde{G}}) = \mathrm{mr}(\mathring{\overline{G_1}} \underset{x}{\oplus} K_2) + \mathrm{mr}(\mathring{\overline{G_2}} \underset{y}{\oplus} K_2).$$

Since $n(\mathring{\widetilde{G}}) = n(\mathring{\overline{G_1}}) + n(\mathring{\overline{G_2}}) + 1 = n(\mathring{\overline{G_1}} \underset{x}{\oplus} K_2) - 1 + n(\mathring{\overline{G_2}} \underset{y}{\oplus} K_2) - 1 + 1$, then

$$n(\mathring{\widetilde{G}}) - \mathrm{mr}(\mathring{\widetilde{G}}) = n(\mathring{\overline{G_1}} \underset{x}{\oplus} K_2) - \mathrm{mr}(\mathring{\overline{G_1}} \underset{x}{\oplus} K_2) + n(\mathring{\overline{G_2}} \underset{y}{\oplus} K_2) - \mathrm{mr}(\mathring{\overline{G_2}} \underset{y}{\oplus} K_2) - 1,$$

which is equivalent to the desired equation. □

In the above lemma we have used the cut-vertex reduction formula. The proof of this result is constructive and preserves universal optimality for the matrices that we consider (see [6, Theorem 2.19]).

THEOREM 3.3. *For every connected graph G and field F,*

$$\mathrm{M}(F, \mathring{\widetilde{G}}) = \mathrm{Z}(\mathring{\widetilde{G}}) = m(G) - n(G) + 1 + \mathrm{Z}(\mathring{\overline{BF(G)}})$$

and $\overset{\circ}{\overline{G}}$ *has a universally optimal matrix.*

Proof. We proceed by induction. If G is the graph on a single vertex, then the formula gives 1 establishing the base case.

Now suppose that the result holds for all connected graphs on fewer than n vertices, and consider a connected graph on n vertices. If each bridge in the graph is incident to a leaf, then G is a single island with some pendent vertices and this result was handled in Theorem 3.1. So we may assume that there is a bridge that is not incident to a leaf.

Let $e = \{x, y\}$ denote this bridge, so that the graph consists of component G_1 with vertex x, component G_2 with vertex y, and e joining x and y. Now consider the graphs $H_1 = G_1 \underset{x}{\oplus} K_2$ and $H_2 = G_2 \underset{y}{\oplus} K_2$. We note that $m(G) = m(H_1) + m(H_2) - 1$ and $n(G) = n(H_1) + n(H_2) - 2$. Also by assumption neither G_1 nor G_2 is a single vertex, and so both H_1 and H_2 are connected graphs with fewer than n vertices.

We now have

$$
\begin{aligned}
\mathrm{M}(\overset{\circ}{\overline{G}}) &= \mathrm{M}(\overset{\circ}{\overline{G_1}} \underset{x}{\oplus} K_2) + \mathrm{M}(\overset{\circ}{\overline{G_2}} \underset{y}{\oplus} K_2) - 1 \\
&= \mathrm{M}(\overset{\circ}{\overline{G_1 \underset{x}{\oplus} K_2}}) + \mathrm{M}(\overset{\circ}{\overline{G_2 \underset{y}{\oplus} K_2}}) - 1 \\
&= \mathrm{M}(\overset{\circ}{\overline{H_1}}) + \mathrm{M}(\overset{\circ}{\overline{H_2}}) - 1 \\
&= \big(m(H_1) - n(H_1) + 1 + \mathrm{Z}\big(\overset{\circ}{\overline{BF(H_1)}}\big)\big) + \\
&\qquad \big(m(H_2) - n(H_2) + 1 + \mathrm{Z}\big(\overset{\circ}{\overline{BF(H_2)}}\big)\big) - 1 \\
&= \big(m(G) + 1\big) - \big(n(G) + 2\big) + 2 + \mathrm{Z}\big(\overset{\circ}{\overline{BF(H_1)}}\big) + \mathrm{Z}\big(\overset{\circ}{\overline{BF(H_2)}}\big) - 1 \\
&= m(G) - n(G) + \mathrm{Z}\big(\overset{\circ}{\overline{BF(H_1)}}\big) + \mathrm{Z}\big(\overset{\circ}{\overline{BF(H_2)}}\big).
\end{aligned}
$$

The first line is an application of Lemma 3.2, while the second line follows by noting that adding a pendent vertex to a pendent vertex does not change the maximum nullity of a graph, nor the property of having a universally optimal matrix. The remainder reduces to substituting in the above information, using the induction hypothesis on H_1 and H_2, and simplifying the result.

To conclude it suffices to show that

$$\mathrm{Z}\big(\overset{\circ}{\overline{BF(H_1)}}\big) + \mathrm{Z}\big(\overset{\circ}{\overline{BF(H_2)}}\big) = 1 + \mathrm{Z}\big(\overset{\circ}{\overline{BF(G)}}\big).$$

If we take an optimal set of zero forcing paths for $\mathrm{Z}\big(\overset{\circ}{\overline{BF(G)}}\big)$, then the vertex corresponding to e will only be involved in a single zero forcing path. So we can use the same zero forcing paths on H_1 and H_2 that we used for G where we might need to break up one path (i.e., increase the total by one), thus the left hand side is at most the right hand side.

On the other hand we can take an optimal set of zero forcing paths for $\mathrm{Z}\big(\overline{BF(H_1)}\big)$ and $\mathrm{Z}\big(\overline{BF(H_2)}\big)$ where we insist that one of the zero forcing paths must end at the pendent vertex we have added to G_1 and that one of the zero forcing paths must start at the pendent vertex we have added to G_2 (note for a zero forcing set Z, a pendent vertex must be in Z or Z-terminal, and these two properties can be interchanged by reversing the zero forcing process [2, Theorem 2.6]). We can now combine the two sets of forcing paths and glue two forcing paths together (reducing the total by one). Thus we can conclude that the right hand side is at most the left hand side.

This establishes the equality and concludes the proof. □

Remark 3.4. By Theorem 3.3, $\mathrm{Z}(\overleftrightarrow{G}) = m(G) - n(G) + 1 + \mathrm{Z}(\overline{BF(G)})$, and so the construction in Theorem 2.6 gives a minimum zero forcing set. In fact, if G is 2-edge connected, every minimum zero forcing set of $\overleftrightarrow{G}$ must contain exactly one original vertex, which can be chosen arbitrarily, the remainder being edge-vertices. To see this, if $\overleftrightarrow{G}$ had a zero forcing set of size $m(G) - n(G) + 2$ with two or more original vertices, say u and v, then there is a zero forcing process so that some original vertex w is never used to force (i.e., either the last vertex forced is an original vertex and this is w or the last vertex forced is an edge vertex and the neighbor of the edge vertex that did not force it is w). Now construct a new graph G' by adding pendent vertices to u, v, and w, so that $BF(G') = K_{1,3}$. Then there is a zero forcing set for $\overleftrightarrow{G'}$ of size $m(G') - n(G') + 2$, i.e., use the zero forcing set of $\overleftrightarrow{G}$ given above and replace the vertices u and v by the pendent vertices we added adjacent to them. Now forcing as before we will end at w, which can force out its pendent vertex. But this is impossible since Theorem 3.3 shows that the minimum zero forcing set of G' has size $m(G') - n(G') + 1 + \mathrm{Z}(\overline{BF(G')}) > m(G') - n(G') + 2$ because $\mathrm{Z}(\overline{BF(G')}) = \mathrm{Z}(BF(G')) = 2$.

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