The principal rank characteristic sequence over various fields

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Abstract

Given an $n \times n$ matrix, its principal rank characteristic sequence is a sequence of length $n+1$ of 0s and 1s where, for $k = 0, 1, \ldots, n$, a 1 in the kth position indicates the existence of a principal submatrix of rank k and a 0 indicates the absence of such a submatrix. The possible principal rank characteristic sequences for symmetric matrices over various fields, including $\mathbb{Z}_2$ and complex matrices are investigated, with all such sequences determined for all n over any field with characteristic 2.

1 Introduction

Given an $n \times n$ symmetric matrix A over some field $\mathbb{F}$ the *principal rank characteristic sequence* of A (abbreviated pr-sequence or $\text{pr}(A)$) is defined as $\text{pr}(A) = r_0]r_1r_2 \cdots r_n$ where

$$r_k = \begin{cases} 1 & \text{if } A \text{ has a principal submatrix of rank } k; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $r_0 = 1$ if and only if A has a 0 diagonal entry. Brualdi et al. [2] introduced the definition of a pr-sequence for a real symmetric matrix as a simplification of the principal minor assignment problem as stated in [5]; see also [7]. In [2] there is also mention of the

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case $\mathbb{F} = \mathbb{C}$ and the Hermitian matrix case. Note that here we denote a pr-sequence by $r_0]r_1r_2\cdots r_n$ (rather than by $r_0r_1r_2\cdots r_n$ as in [2]) to visually emphasize the special nature of r_0.

We use the following result to determine the rank, and hence to work with pr-sequences. Here $A[S|T]$ denotes the submatrix of A on rows indexed by the set S and columns indexed by the set T. If $S = T$, then we write $A[S]$ for the principal submatrix lying in rows and columns S.

Theorem 1.1. *If $A \in \mathbb{F}^{n\times n}$ is symmetric, or $A \in \mathbb{C}^{n\times n}$ is Hermitian, then* $\operatorname{rank} A = \max\{|S| : \det(A[S]) \neq 0\}$.

Proof. This is immediate from [4, Corollary 8.9.2] for symmetric matrices, and for $A \in \mathbb{C}^{n\times n}$ Hermitian it follows from the equality of algebraic and geometric multiplicity of the eigenvalue zero. □

All matrices in this paper are square, and unless specified otherwise all matrices are symmetric. We are interested in which pr-sequences are *attainable*, i.e., can be attained by some desired matrix, and also which sequences are *forbidden*, i.e., no desired matrix attains the sequence. The case $\mathbb{F} = \mathbb{R}$ was studied by Brualdi et al. [2], and in this paper we continue the investigation into pr-sequences by considering the problem over different fields (Sections 2 and 3) and extending the results of [2] over $\mathbb{R}$ (Section 4). In particular, in Section 3 we identify all attainable pr-sequences of all orders over any field with characteristic 2. For some results we use the $(0, 1)$ adjacency matrix of a graph G, denoted by $A(G)$, and in Section 5 we give results for pr-sequences of such matrices with full rank.

2 Basic facts about pr-sequences

In this section we discuss basic facts about principal rank characteristic sequences over various fields, and highlight some sequences that are forbidden, as well as indicate examples of sequences that are always attainable. We let $\overline{r_i\cdots r_j}$ indicate that the (complete) sequence may be repeated as many times as desired (or omitted entirely).

2.1 Pr-sequences forbidden over all fields

1. $0]0\cdots$ is undefined (forbidden by definition), and $1]1$ is also undefined for order 1.

2. $1]r_1 0\cdots 1$ is forbidden for symmetric matrices over all fields and for Hermitian matrices [2, Theorem 4.1].

3. $\cdots 001\cdots$ is forbidden for symmetric matrices over all fields as well as for Hermitian matrices.

Statement 2 can be seen by noting that there is a zero on the diagonal ($r_0 = 1$) and this zero must in turn force the corresponding row and column where it lies to be zero ($r_2 = 0$), and so the matrix cannot have full rank ($r_n \neq 1$).

Statement 3 was established for $\mathbb{F} = \mathbb{R}$ in [2, Theorem 4.4], but we give here a simpler more general proof.

Theorem 2.1. *The sequence $\cdots 001 \cdots$ is forbidden for symmetric matrices over any field and for Hermitian matrices.*

Proof. Let $A \in \mathbb{F}^{n\times n}$ be symmetric or $A \in \mathbb{C}^{n\times n}$ be Hermitian, and suppose $\operatorname{pr}(A) = r_0]r_1 \cdots r_n$ with $r_k = r_{k+1} = 0$. Let B be a $(k+2)\times(k+2)$ principal submatrix of A, and C be a $(k+1)\times(k+1)$ principal submatrix of B. By Theorem 1.1, $\operatorname{rank} C$ is the maximum order of a nonzero principal minor. Since any principal minor of C is a principal minor of A and $r_k = r_{k+1} = 0$, then $\operatorname{rank} C \le k-1$. Since B is obtained from C by adding one row and one column, $\operatorname{rank} B \le \operatorname{rank} C + 2 \le k+1$. Thus every $(k+2)\times(k+2)$ principal submatrix of A is singular, implying that $r_{k+2} = 0$. □

2.2 Pr-sequences attainable over all fields

In the case $n = 1$, by definition the only attainable sequences over any field are $0]1$ and $1]0$. From now on we assume that $n \ge 2$, and give some pr-sequences for general n that can be attained over any field.

1. $1]00\overline{0}$ is attained by the $n \times n$ zero matrix O_n.

2. $0]11\overline{1}$ is attained by the $n \times n$ identity matrix I_n.

3. $1]11\overline{1}$ is attained by $L_2 \oplus I_{n-2}$ where $L_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

4. $1]01\overline{01}$ for n even, $1]010\overline{10}$ for n odd is attained by $A(P_n)$, where P_n is the path on n vertices.

2.3 Field dependent pr-sequences

The following pr-sequences can be attained over some fields, but not all. See Theorem 3.1 for all attainable sequences over a field of characteristic 2.

1. $1]011$ is attainable over $\mathbb{R}$ by $A(K_3)$ where K_3 is the complete graph on 3 vertices, but is not attainable over a field with characteristic 2.

2. $0]101\overline{01}$ for n odd is attained by $(A(C_n))^{-1}$ for any field with characteristic not 2 where C_n is the cycle graph on n vertices (see [2, Theorem 2.7] for the real field); it is not attainable over a field with characteristic 2.
   $0]10\overline{10}10$ (n even) is attained by appending 0 to sequence $0]101\overline{01}$ of length $n-1$ [2, Theorem 2.6]; it is not attainable over a field with characteristic 2.

3. $1]01101$ is forbidden for real symmetric matrices [2, Theorem 6.3] but attainable for complex symmetric [2, Example 6.8] and Hermitian matrices [2, p. 2151]. Furthermore, any pr-sequence that contains 101101 as a subsequence is forbidden for real symmetric matrices [2, Theorem 6.4].

4. The pr-sequence $0]110101\cdots$ is forbidden for real symmetric matrices [2, Theorem 7.2] and complex symmetric matrices by a similar argument, but $0]110101$ is attainable for Hermitian matrices, as shown by the next example.

Example 2.2. Let

$$A = \begin{bmatrix} 0 & i & i & 1 & 0 & 0 \\ -i & 0 & i & 0 & i & 0 \\ -i & -i & 0 & 0 & 0 & 1+i \\ 1 & 0 & 0 & 0 & i & i \\ 0 & -i & 0 & -i & 0 & i \\ 0 & 0 & 1-i & -i & -i & 0 \end{bmatrix}.$$

Then $\mathrm{pr}(A) = 1]010111$, and $B = A^{-1}$ has $\mathrm{pr}(B) = 0]110101$, which can be verified by computing the minors.

The above examples show that the attainable pr-sequences for real, complex symmetric, and Hermitian matrices are all different. Another field to consider is the rational numbers, and it is an open question whether the pr-sequences that are attainable for rationals differ from those attainable for reals. One possible candidate for such a difference is 1]0111101, which by an exhaustive computer search is not attainable for any adjacency matrix, but is attainable for a real matrix with coefficients that come from roots of a particular cubic (see the construction in [2, Example 6.7]). This sequence beginning with 1]0 answers negatively an open question posed in [2] since it is achievable over the reals but not by the adjacency matrix of any graph. From results of [2], seven is the smallest order for such an example.

2.4 Forming pr-sequences

The following facts give generic information about some pr-sequences that are attainable, and useful tools to extend, reverse, or to combine pr-sequences over all fields. They are proved over $\mathbb{R}$ in [2, Theorems 2.3, 2.6, 2.7].

1. If $\mathrm{pr}(A) = r_0]r_1 \cdots r_n$, then $\mathrm{pr}(A) = 1]r_1 \cdots r_n 0$ is attained by $A \oplus O_1$.

2. If $\mathrm{pr}(A) = r_0]r_1 \cdots r_n$, then $\mathrm{pr}(A) = r_0]r_1 \cdots r_n 0$ is attained by duplicating a row and column of A. In particular, appending 0 to an attainable sequence results in another attainable sequence.

3. (Inverse Palindrome Theorem). Suppose $A \in \mathbb{F}^{n\times n}$ is an $n \times n$ nonsingular matrix with $\mathrm{pr}(A) = r_0]r_1 \cdots r_{n-1}1$. Let $\mathrm{pr}(A^{-1}) = r'_0]r'_1 \cdots r'_{n-1}1$. Then $r'_i = r_{n-i}$ for each i with $1 \le i \le n-1$, and $r'_0 = 1$ if and only if A has some principal minor of order $n-1$ that is zero. This is established using Jacobi's identity; see, e.g., [6, p. 24].

Statement 2 shows that an attainable sequence can be extended by 0. However it should be noted that if an attainable sequence ends with 0, then the ending 0 cannot always be dropped to realize another attainable sequence. As an example of this, 1]101 is forbidden over all fields (see Subsection 2.1), but 1]1010 is attainable over $\mathbb{R}$ by $(J_3 - 2I_3) \oplus O_1$, where J_n is the $n \times n$ matrix of all ones. In fact 1]1010 is attainable over a field if and only if its characteristic is not 2.

Let $\mathrm{supp}(A) = \{i : r_i = 1\}$; observe that $\mathrm{supp}(A)$ uniquely determines the pr-sequence and vice-versa. Given two sets S and T, define $S + T = \{s + t : s \in S, t \in T\}$. Then we have the following useful general result for a direct sum of two matrices.

Theorem 2.3. (Reducible Matrix Theorem) *If A, B are square Hermitian or symmetric over a field $\mathbb{F}$, then*

$$\operatorname{supp}(A \oplus B) = \big(\operatorname{supp}(A) + \operatorname{supp}(B)\big) \cup \operatorname{supp}(A) \cup \operatorname{supp}(B).$$

Proof. The principal submatrices of $A \oplus B$ can be grouped into three families: ones that only use submatrices from A, ones that only use submatrices from B, and ones that use submatrices from both A and B. For the first family, if this submatrix has full rank in A, then it is also has full rank in $A \oplus B$. Therefore if $s \in \operatorname{supp}(A)$, then $s \in \operatorname{supp}(A \oplus B)$. For the second family, a similar argument shows that if $t \in \operatorname{supp}(B)$, then $t \in \operatorname{supp}(A \oplus B)$. In the third family, a submatrix has the form $A' \oplus B'$ where A' and B' are submatrices of A and B, respectively. The principal submatrix corresponding to $A' \oplus B'$ has full rank if and only if the principal submatrices corresponding to A' and B' have full rank. In particular, if A' has full rank and order s and B' has full rank and order t, then $s \in \operatorname{supp}(A), t \in \operatorname{supp}(B)$ implies that $s + t \in \big(\operatorname{supp}(A) + \operatorname{supp}(B)\big)$. □

3 Pr-sequences over a field with characteristic 2

The smallest field is $\mathbb{Z}_2$ (the integers modulo 2), and Subsection 2.3 has some examples showing that pr-sequences over this field differ from those over $\mathbb{R}$. Because $\mathbb{Z}_2$ is a finite field, we are able to do exhaustive computer searches for small values of n to determine all attainable pr-sequences. Table 1 gives the results for $n \leq 5$.

Table 1: Attainable pr-sequences over $\mathbb{Z}_2$ for $n \leq 5$.

$n = 2$	$n = 3$	$n = 4$	$n = 5$
1]00	1]000	1]0000	1]00000
1]01	1]010	1]0100	1]01000
0]10	0]100	1]0101	1]01010
0]11	0]110	0]1000	0]10000
1]10	0]111	0]1100	0]11000
1]11	1]100	0]1110	0]11100
	1]110	0]1111	0]11110
	1]111	1]1000	0]11111
		1]1100	1]10000
		1]1110	1]11000
		1]1111	1]11100
			1]11110
			1]11111

This table is highly suggestive about attainable pr-sequences, and in this section we identify all attainable pr-sequences for matrices of all orders over any field with characteristic 2.

Theorem 3.1. *For $n \geq 2$ over a field with characteristic 2, a pr-sequence is attainable if and only it it is of one of the following forms*

$$1]\overline{01}\ \overline{0}, \qquad 0]1\ \overline{1}\ \overline{0}, \qquad 1]1\ \overline{1}\ \overline{0}.$$

Namely, attainable sequences have one of the following forms:
1. 1 *followed by a (possibly empty) sequence of* 01*s with a (possibly empty) sequence of* 0*s at the end;*
2. 0 *followed by a sequence of at least one* 1 *followed by a (possibly empty) sequence of* 0*s;*
3. a sequence of at least two 1*s followed by a (possibly empty) sequence of* 0*s.*

Before proving this theorem we give two lemmas. The first shows that in a field with characteristic 2 all terms in the pr-sequence except possibly r_0 are preserved under congruence.

Lemma 3.2. *Let* $\mathbb{F}$ *be a field with characteristic* 2*, let* A *be an* $n \times n$ *symmetric matrix over* $\mathbb{F}$ *with* $\operatorname{pr}(A) = r_0]r_1 \dots r_n$*, and let* E *be an* $n \times n$ *invertible matrix over* $\mathbb{F}$*. Then* $\operatorname{pr}(EAE^T) = r_0']r_1' \dots r_n'$ *where* $r_i' = r_i$ *for* $i = 1, \dots, n$*.*

Proof. Without loss of generality, E is an elementary row operation matrix, and the result is immediate except in the case where E adds a multiple of one row to another, without loss of generality adding m times row $n-1$ to row n. That is, it suffices to consider the case

$$E = I_{n-2} \oplus \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}.$$

By the invertibility of E it suffices to show, for an arbitrary integer k in the range $1 \le k \le n$, that if every $k \times k$ principal submatrix of A has determinant 0, then every $k \times k$ principal submatrix of EAE^T also has determinant 0.

Congruence by the chosen E only affects the determinants of principal submatrices on index sets including row and column n but not including row or column $n-1$. Accordingly, let S be a subset of $\{1, \dots, n-2\}$ of cardinality $k-1$, and define a function M from matrices of order n to matrices of order 2 as follows:

$$M(A, S) = \begin{bmatrix} \det A[S \cup \{n-1\}] & \det A[S \cup \{n-1\} | S \cup \{n\}] \\ \det A[S \cup \{n\} | S \cup \{n-1\}] & \det A[S \cup \{n\}] \end{bmatrix}.$$

By the multilinearity of the determinant, if

$$M(A, S) = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

then

$$M(EA, S) = \begin{bmatrix} a & b \\ b + ma & c + mb \end{bmatrix}$$

and

$$\begin{aligned} M(EAE^T, S) &= \begin{bmatrix} a & b + ma \\ b + ma & c + 2mb + m^2 a \end{bmatrix} \\ &= \begin{bmatrix} a & b + ma \\ b + ma & c + m^2 a \end{bmatrix} \text{ (in characteristic 2).} \end{aligned}$$

In particular, if $r_k = 0$ then $a = 0$ and $c = 0$, and by the generality of S every principal submatrix of order k in EAE^T has determinant 0 as well. □

The second lemma, a variation of a well-known result (see, for example, [3, page 426]), is a canonical form under congruence for symmetric matrices over a field with characteristic 2.

Lemma 3.3. *Let A be a symmetric matrix over a field $\mathbb{F}$ with characteristic 2. Then A is congruent to the direct sum of a (possibly empty) invertible diagonal matrix D, a (possibly empty) direct sum of $A(K_2)$ matrices, and a (possibly empty) zero matrix.*

Note that if $\mathbb{F}$ is a finite field with characteristic 2, then the matrix D can be taken to be an identity matrix, but not in general over an infinite field because in that case there can be elements of $\mathbb{F}$ that are not squares.

Example 3.4. Over $\mathbb{Z}_2$, let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $EAE^T = I_1 \oplus A(K_2)$. Here $\text{pr}(EAE^T) = 1]111 = \text{pr}(A)$.

Proof of Theorem 3.1. First we show that each sequence in the theorem statement can be attained. For this consider the following matrices.

- $A(K_2) \oplus A(K_2) \oplus \cdots \oplus A(K_2) \oplus O_{n-2k}$, i.e., the adjacency matrix for the graph consisting of $k \geq 0$ disjoint edges and $n - 2k$ isolated vertices. This has a full rank principal submatrix of order $2, 4, \ldots, 2k$ and every other order of principal submatrix corresponds to a graph that has at least one isolated vertex and so its adjacency matrix does not have full rank. Thus its pr-sequence is the first one given in the theorem statement.

- $I_{k-1} \oplus J_{n-k+1}$ for $1 \leq k \leq n$ has pr-sequence $0]11\cdots100\cdots0$ with k consecutive 1s, i.e., there is an identity matrix of order $\leq k$ as a principal submatrix; anything larger must have a repeated row/column. This gives the second sequence in the statement.

- $I_k \oplus O_{n-k}$ for $1 \leq k < n$ has pr-sequence $1]1\cdots100\cdots0$, with $r_j = 0$ for $j > k$. For the case $1]1\cdots1$, use the matrix $L_2 \oplus I_{n-2}$; see Section 2.2.

Now assume that a certain sequence is attainable and let A be an $n \times n$ matrix that attains it. Then by Lemma 3.3, A is congruent to a matrix B that is the direct sum of an invertible diagonal matrix, several copies of $A(K_2)$ and a 0 matrix (some of these summands may be empty). There are then two cases to consider.
Case 1: The diagonal summand is empty so that $B = A(K_2) \oplus \cdots \oplus A(K_2) \oplus O_\ell$ for $0 \leq \ell \leq n$, where there are k copies of $A(K_2)$ and $\ell + 2k = n$.
By Lemma 3.2, $r_i(A) = r_i(B)$ for $i = 1, \ldots, n$. Since $r_1(A) = r_1(B) = 0$, every diagonal entry of A is 0, and $r_0(A) = 1 = r_0(B)$. Thus $\text{pr}(A) = \text{pr}(B) = 1]0101\cdots010\cdots0$, with the rightmost 1 in r_{2k}.
Case 2: $B = D \oplus A(K_2) \oplus \cdots \oplus A(K_2) \oplus O_\ell$ where D is an invertible diagonal matrix of order $j \geq 1$, there are k copies of $A(K_2)$ and $j + 2k + \ell = n$. In this case $\text{pr}(A) = \text{pr}(B) = r_0]1\cdots10\cdots0$ where r_0 can be 0 or 1 (0 if and only if $k = \ell = 0$), and the rightmost 1 is in r_{j+2k}.
Therefore, only the pr-sequences listed in the theorem statement are attainable. □

Note that in a field with characteristic 2, r_0 in the pr-sequence need not be preserved under congruence, as the next example shows.

Example 3.5. Suppose that $\mathbb{F}$ is a field with characteristic 2. Then

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbb{F}^{2\times 2}$$

has $\mathrm{pr}(A) = 1]11$. Taking

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

gives $EAE^T = I_2$ and $\mathrm{pr}(I_2) = 0]11$.

In a finite field with characteristic 2 [8, Lemma 3] shows that if A is an $n \times n$ matrix with $\mathrm{pr}(A) = 1]1r_2 \ldots r_n$, then there exists an invertible matrix E so that $\mathrm{pr}(EAE^T) = 0]1r_2 \ldots r_n$. Table 1 illustrates this for $n = 2, \ldots, 5$.

Also note that in any field that does not have characteristic 2, the pr-sequence $0]101$ can be attained, as the next example shows.

Example 3.6. The matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

has pr-sequence $0]101$ over any field of characteristic not 2 ($\det A = -4$).

An interesting problem is to determine what happens for other finite fields. For example, by an exhaustive search for the field $\mathbb{Z}_3$ every pr-sequence for matrices of order at most 6 that is not forbidden over all fields is attainable *except* for $0]11011$, $0]110101$, $0]110110$ and $0]110111$. Motivated by this, we have the following result.

Proposition 3.7. *Over the finite field $\mathbb{Z}_3$, any sequence that begins with $0]1101$ is of the form $0]1101\overline{0}$.*

Proof. Suppose that for some matrix A over $\mathbb{Z}_3$, its pr-sequence begins with $0]1101$ and $r_k = 1$ for $k \geq 5$. If $r_5 = 1$, then there is a full rank 5×5 principal submatrix, call it B. The matrix B inherits $r_0 = 0$ and $r_3 = 0$ (in general 0s are always inherited when taking submatrices as well as 1s on both sides where applicable, i.e., to avoid 001), and so B must have the pr-sequence $0]11011$, which is ruled out by an exhaustive search. Now suppose $r_5 = 0$ and $r_6 = 1$, then by the same argument there is some 6×6 principal submatrix that has the pr-sequence $0]110101$, but this has also been ruled this out by an exhaustive search. Therefore $r_5 = 0$ and $r_6 = 0$ and so $r_k = 0$ for all $k \geq 5$, which is a contradiction. □

Whether for matrices of order ≥ 7 over $\mathbb{Z}_3$ there are any forbidden sequences or subsequences in addition to those ruled out by Proposition 3.7 and those forbidden over all fields is unknown. Similarly, what happens over larger finite fields is also unknown.

4 Pr-sequences for order 7 symmetric matrices over $\mathbb{R}$

Over the real number field the problem of determining which sequences are attainable for order up through 6 was solved in [2]. We now determine all the pr-sequences of order 7 that can be attained by real symmetric matrices. The results are summarized in two tables. Table 2 covers pr-sequences that cannot be attained by listing forbidden subsequences for real matrices, as established in [2], and two additional sequences that cannot occur. Table 3 lists all pr-sequences that can be attained except those of the form $r_0]r_1r_2r_3r_4r_5r_60$ where $r_0]r_1r_2r_3r_4r_5r_6$ is attainable. We show that this covers all order 7 pr-sequences.

Table 2: Forbidden (sub)sequences for real matrices. (Note that some of the $\cdots$ may be empty.)

Forbidden (sub)sequences	Justification
$\cdots 001 \cdots$	Section 2.1 Statement 3
$\cdots 101101 \cdots$	Section 2.3 Statement 3
$0]110101\cdots$	Section 2.3 Statement 4
$1]10\cdots 1$	Section 2.1 Statement 2
$0]1010111$	Proposition 4.1
$1]1101011$	Proposition 4.1

Proposition 4.1. *The two sequences* $0]1010111$ *and* $1]1101011$ *are not attainable by real symmetric matrices.*

Proof. To show $0]1010111$ is not attainable over $\mathbb{R}$, by [2, Proposition 8.1] the problem reduces to showing by an exhaustive computer search that this cannot occur for a ± 1 matrix with 1s on the diagonal and in every entry in the first row and column.

To show $1]1101011$ is not attainable over $\mathbb{R}$, first note that if there were such a matrix A, then $\mathrm{pr}(A^{-1}) = *]1010111$ for some $*$. However, $* = 0$ gives the sequence just ruled out. On the other hand $* = 1$ gives the sequence $1]10\cdots 1$, which is forbidden by Table 2. □

Ignoring the undefined pr-sequences that begin $0]0$, there are $192 = 3 * 2^6$ pr-sequences of order 7 to consider. The following rules from Table 2 have been applied to systematically classify pr-sequences; supporting documentation is given in [1].

- The subsequence $\cdots 001 \cdots$ is forbidden over any field, and eliminates 105 pr-sequences, leaving 87 pr-sequences.
- The subsequence $\cdots 101101 \cdots$ is forbidden for real matrices. This subsequence eliminates an additional 11 pr-sequences, leaving 76 pr-sequences.
- The two pr-sequences $0]1101010$ and $0]1101011$ are forbidden for real matrices, leaving 74 pr-sequences.

- The family of forbidden pr-sequences $1]10\cdots 1$ eliminates an additional 4 pr-sequences, leaving 70.

- The two additional pr-sequences $0]1010111$ and $1]1101011$ are forbidden by Proposition 4.1, leaving 68 attainable pr-sequences of order 7.

Thus a total of 124 (defined) pr-sequences of order 7 are unattainable.

It remains to show that each of the remaining 68 pr-sequences is attainable. As established in [2] (see Section 2.4 Statement 2), if any sequence $r_0]r_1\cdots r_n$ is attainable, then $r_0]r_1\cdots r_n0$ is also attainable. Since the 46 pr-sequences that are attainable for $n=6$ are listed in [2] (in Table 7.1 and in Tables 5.1-5.4, 6.1 by appending 0s), these can be used to find 46 attainable pr-sequences for $n=7$ by appending a 0. These pr-sequences are identified in the supporting documentation [1], together with the least n in which the initial subsequence of order n is attainable; see also Table 4 below. These pr-sequences obtained by appending a 0 to an order 6 attainable sequence are omitted from Table 3, which lists in lexicographic order the 22 remaining pr-sequences attainable by real symmetric matrices of order 7, and matrices realizing these sequences.

Here we give an overview of the methods used to find these matrices. We conducted a computer search of pr-sequences of adjacency matrices of graphs. This search found that, with the exception of the pr-sequence $1]0111101$ that is attained by a circulant matrix constructed in [2, Example 6.7], every other order 7 pr-sequence beginning with $1]0$ that does not have any of the forbidden subsequences and is not in the form of an order 6 attainable pr-sequence with a 0 appended is attained by an adjacency matrix; two of the graphs used are shown in Figure 1. Interestingly, G_2 is the only 7-vertex graph whose adjacency matrix attains the pr-sequence $1]0111011$. The Inverse Palindrome Theorem (Section 2.4 Statement 3), the Reducible Matrix Theorem (Theorem 2.3), and additional results in [2] were also used to construct matrices. Certain pr-sequences beginning $1]1$ are attainable by matrices of the following form $Q_{7,k}(G)$ where

$$Q_{7,k}(G) = \begin{bmatrix} 2k & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & & & & & & \\ 1 & & & & & & \\ 1 & & & A(G) & & & \\ 1 & & & & & & \\ 1 & & & & & & \\ 1 & & & & & & \end{bmatrix}$$

(where as usual $A(G)$ is the adjacency matrix of the graph G); for these matrices used in the table, the graph G is specified. This notation extends that of [2, Theorem 3.7]; we denote the matrix $Q_{n,k}$ of [2] by $Q_{n,k}(kK_2)$ (where kG means the disjoint union of k copies of G). The fact that each matrix produces the claimed pr-sequence has been verified computationally.

Theorem 4.2. *There are exactly* 68 *attainable pr-sequences of order* 7*, namely the* 22 *sequences listed in Table 3 and the* 46 *sequences of the form* $r_0]r_1r_2r_3r_4r_5r_60$ *with* $r_0]r_1r_2r_3r_4r_5r_6$ *attainable.*

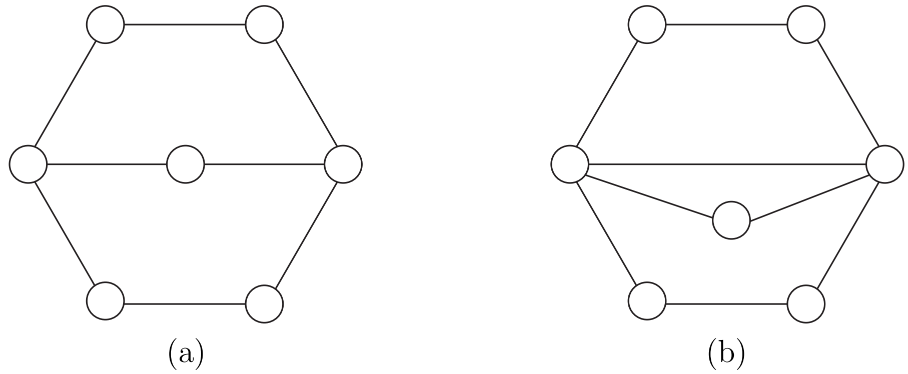


Figure 1: The graphs (a) G_1, which is C_6 with a subdivided equitable chord, and (b) G_2, which is G_1 with an edge between the two degree 3 vertices.

Table 3: All pr-sequences for order 7 that can be attained by real symmetric matrices except those of the form $r_0]r_1r_2r_3r_4r_5r_60$ with $r_0]r_1r_2r_3r_4r_5r_6$ attainable. The sequences are listed in lexicographic order.

pr-Sequence	Real matrix	Comments
0]1010101	$(A(C_7))^{-1}$	
0]1011101	$(A(G_2))^{-1}$	G_2 is the graph in Figure 1(b)
0]1011111	$J_7 - 2I_7$	
0]1101111	$J_7 - 3I_7$	
0]1110101	$(A(G_1))^{-1}$	G_1 is the graph in Figure 1(a)
0]1110111	$J_7 - 4I_7$	
0]1111011	$J_7 - 5I_7$	
0]1111101	$J_7 - 6I_7$	
0]1111111	I_7	
1]0101011	$A(C_7)$	
1]0101111	$A(G_1)$	G_1 is the graph in Figure 1(a)
1]0111011	$A(G_2)$	G_2 is the graph in Figure 1(b)
1]0111101	C	C is the circulant matrix in [2, Example 6.7]
1]0111111	$J_7 - I_7$	$= A(K_7)$
1]1010110	$M \oplus O_1$	$M = M_{0101011}$ in [2, p. 2153]
1]1011110	$(J_6 - 2I_6) \oplus O_1$	
1]1101111	$Q_{7,1}(3K_2)$	[2, Theorem 3.7]
1]1110101	$(A(C_5))^{-1} \oplus A(K_2)$	
1]1110111	$(Q_{7,1}(P_4 \dot{\cup} K_2))^{-1}$	
1]1111011	$Q_{7,2}(3K_2)$	[2, Theorem 3.7]
1]1111101	$(A(K_3))^{-1} \oplus A(K_2) \oplus A(K_2)$	
1]1111111	$L_2 \oplus I_5$	$L_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

In both [2] and in Table 3, for each order only the *new* pr-sequences of order n (e.g. for order 7, those not of the form $r_0]r_1r_2r_3r_4r_5r_60$ with $r_0]r_1r_2r_3r_4r_5r_6$ attainable), are listed. It is of interest to collect the actual number of attainable pr-sequences for each order, and this is done in Table 4, with data taken from [2] and Theorem 4.2. The total number of (defined) pr-sequences of order n for $n \geq 3$ is computed by the formula $3 * 2^{n-1}$, because there are 3 choices for $r_0]r_1$, namely $0]1, 1]0, 1]1$, and 2 choices for each $r_i, i = 2, \ldots, n$. We observe that the fraction of (defined) pr-sequences that are attained is declining as n increases, which raises an interesting question in its own right.

Table 4: Number of pr-sequences attained and fraction of (defined) pr-sequences attained for $n \leq 7$.

n	new	# attained	total #	% attained
1	2	2	2	100%
2	4	6	6	100%
3	4	10	12	83%
4	8	18	24	75%
5	11	29	48	60%
6	17	46	96	48%
7	22	68	192	35%

The pr-sequence $0]1010111$ from Proposition 4.1 is not attainable for real symmetric matrices, and also by a similar argument is not attainable for complex symmetric matrices. It is however attainable by a Hermitian matrix, as the following example shows.

Example 4.3. Let $\omega = e^{2\pi i/3}$, i.e., a complex cube root of unity (so $\bar{\omega} = \omega^2$), and consider the Hermitian matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \omega^2 & \omega^2 & \omega & 1 \\ 1 & 1 & 1 & 1 & \omega & \omega & \omega^2 \\ 1 & \omega & 1 & 1 & 1 & \omega^2 & \omega^2 \\ 1 & \omega & \omega^2 & 1 & 1 & 1 & \omega \\ 1 & \omega^2 & \omega^2 & \omega & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega & \omega^2 & 1 & 1 \end{bmatrix}.$$

Since A is Hermitian and every off-diagonal entry has magnitude 1, every 2×2 principal submatrix of A is singular. The principal submatrix $A[2,3,4]$ has full rank. Every 4-tuple in $\{1, \ldots, 7\}$ contains at least one of the following triples:

$$\{1,2,3\}, \{1,2,7\}, \{1,3,4\}, \{1,4,5\}, \{1,5,6\}, \{1,6,7\}, \{2,3,6\},$$

$$\{2,4,5\}, \{2,4,6\}, \{2,5,7\}, \{3,4,7\}, \{3,5,6\}, \{3,5,7\}, \{4,6,7\}.$$

Each of these triples gives a principal submatrix of rank 1, so every 4×4 principal submatrix of A is singular. The 5×5 minors are all equal to 9, the 6×6 minors are all equal to -27, and the determinant of A is 54, giving $\text{pr}(A) = 0]1010111$.

An open question is whether there is a pr-sequence attainable for complex symmetric matrices but not attainable for Hermitian matrices.

5 A curious fact about adjacency matrices

We have previously mentioned that for order 7 there is no adjacency matrix that has pr-sequence 1]0111101; this was established by an exhaustive computer search. This search was extended for the next several orders and below are listed *all* of the attainable pr-sequences for adjacency matrices that have full rank ($r_n = 1$) up through order 9.

Table 5: All pr-sequences attained by (real) full rank adjacency matrices for order ≤ 9.

$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
1]01	1]011	1]0101	1]01011	1]010101	1]0101011	1]01010101	1]010101011
		1]0111	1]01111	1]010111	1]0101111	1]01010111	1]010101111
				1]011101	1]0111011	1]01011101	1]010111011
				1]011111	1]0111111	1]01011111	1]010111111
						1]01110101	1]011101011
						1]01110111	1]011101111
						1]01111101	1]011111011
						1]01111111	1]011111111

In the above table $r_{2k} = 1$ for $k = 1, 2, 3, 4$ as long as the pr-sequence is sufficiently long to contain that entry. For larger graphs a similar result holds.

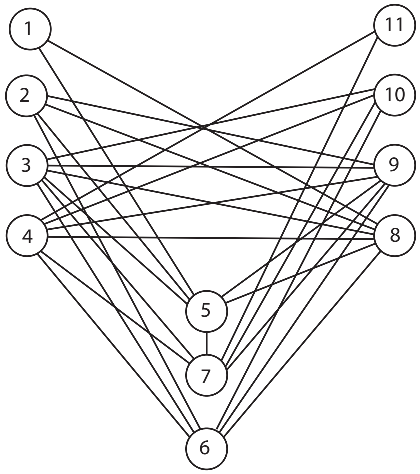
Theorem 5.1. *If A is the adjacency matrix of a graph G with pr(A) having $r_m = 1$ for some $m \geq 10$, then $r_2 = r_4 = r_6 = r_8 = 1$.*

Proof. First note that either $r_8 = 1$ or $r_9 = 1$ (otherwise there would be an occurrence of 001, which is forbidden). Therefore the graph G contains an induced subgraph with a full rank adjacency matrix of order either 8 or 9. In either case using the above table, this subgraph in turn contains induced subgraphs of order 2, 4, 6, and 8 that have full rank adjacency matrices. Since an induced subgraph of an induced subgraph is an induced subgraph the result follows. □

Interestingly, this trend does not continue. In particular, there exist graphs of order 11 with adjacency matrices having $r_{11} = 1$ and $r_{10} = 0$; one such example is now given.

Example 5.2. Figure 2 shows a graph G of order 11 and its adjacency matrix with pr($A(G)$) = 1]01111111101.

For $n = 11$, this graph is one of 15 such graphs out of a count of 728,952,205 that have full rank adjacency matrices.



$$A(G) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 2: A graph with $\mathrm{pr}(A(G)) = 1]01111111101$.

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References

[1] W. Barrett, S. Butler, M. Catral, S. M. Fallat, H. T. Hall, L. Hogben, P. van den Driessche, M. Young. Supporting documentation for order 7 real pr-sequences. Sorted by attainability: `http://orion.math.iastate.edu/lhogben/research/pr_order7_sort.pdf`. Lexicographic order: `http://orion.math.iastate.edu/lhogben/research/pr_order7.pdf`.

[2] R. A. Brualdi, L. Deaett, D. D. Olesky, P. van den Driessche. The principal rank characteristic sequence of a real symmetric matrix. *Linear Algebra and its Applications*, **436**: 2137–2155, 2012.

[3] S. Friedland. Quadratic forms and the graph isomorphism problem. *Linear Algebra and its Applications*, **150**: 423-442, 1991.

[4] C. Godsil and G. Royle. *Algebraic Graph Theory.* Springer-Verlag, New York, 2001.

[5] O. Holtz and H. Schneider. Open problems on GKK $\tau-$matrices. *Linear Algebra and its Applications*, **345**: 263-267, 2002.

[6] R. A. Horn and C. R. Johnson. *Matrix Analysis*, 2nd ed. Cambridge University Press, Cambridge, 2013.

[7] C. R. Johnson. Some outstanding problems in the theory of matrices. *Linear and Multilinear Algebra*, **12**: 99-108, 1982.

[8] J. MacWilliams. Orthogonal matrices over finite fields. *American Mathematical Monthly*, **76**: 152-164, 1969.