The anti-Ramsey number of perfect matching.

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Abstract

An *r-edge coloring* of a graph G is a mapping $h : E(G) \to [r]$, where $h(e)$ is the color assigned to edge $e \in E(G)$. An *exact r-edge coloring* is an r-edge coloring h such that there exists an $e \in E(G)$ with $h(e) = i$ for all $i \in [r]$. Let h be an edge coloring of G. We say G is *rainbow* if no two edges in G are assigned the same color by h. The *anti-Ramsey number*, $AR(G, n)$, is the smallest integer r such that for any exact r-edge coloring of K_n there exists a subgraph isomorphic to G that is rainbow. In this paper we confirm a conjecture of Fujita, Kaneko, Schiermeyer, and Suzuki that states $AR(M_k, 2k) = \max\{\binom{2k-3}{2} + 3, \binom{k-2}{2} + k^2 - 2\}$, where M_k is a matching of size $k \geq 3$.

1 Introduction

An *r-edge coloring* of a graph G is a mapping $h : E(G) \to [r]$, where $h(e)$ is the color assigned to edge $e \in E(G)$. An *exact r-edge coloring* is an r-edge coloring h such that there exists an $e \in E(G)$ with $h(e) = i$ for all $i \in [r]$. Let h be an edge coloring of G. We say G is *rainbow* if no two edges in G are assigned the same color by h. The *anti-Ramsey number*, $AR(G, n)$, is the smallest integer r such that for any exact r-edge coloring of K_n there exists a subgraph isomorphic to G that is rainbow.

The study of anti-Ramsey numbers began with the 1975 paper of Erdős, Simonovits and Sós [1]. In that paper they showed $AR(K_p, n) = t_{p-1}(n) + 2$, where $t_{p-1}(n)$ is the Turán number, and n is sufficiently large. Thirty years later, Montellano-Ballesteros and Neumann-Lara [3] showed this equality holds for all integers n and p such that $n > p \geq 3$.

A *matching* of G is a set of edges in $E(G)$ such that no two have a vertex in common. Let M_k denote the graph consisting of matching of size k, also called a k-matching. Fujita, Kaneko, Schiermeyer, and Suzuki [2] determined $AR(M_k, n) = \max\{\binom{k-2}{2} + (k-2)(n-k+2) + 2, \binom{2k-3}{2} + 2\}$ for $k \geq 2$ and $n \geq 2k+1$. They also give two colorings of K_{2k} which show

$$AR(M_k, 2k) \geq \max\left\{\binom{k-2}{2} + k^2 - 2, \binom{2k-3}{2} + 3\right\}$$

for $k \geq 3$. For the first coloring let A be a subset of $V(K_{2k})$ with size $k+2$. Every edge incident to two vertices in A is colored with color 1, which is never used again. The remaining edges are each colored with a unique color. This is an exact $(\binom{k-2}{2} + k^2 - 3)$-coloring with no rainbow M_k. In the second coloring, consider three vertices $x, y,$ and z of $V(K_{2k})$. The edge yz and all edges incident to x except xy and xz are all colored with color 1, which is never used again. Every edge incident to y or z, but not yz, is colored with color 2, which is never used again. The remaining edges are each colored with a unique color. This is an exact $(\binom{2k-3}{2} + 2)$-coloring with no rainbow M_k.

Theorem 1 (Main Theorem)**.** *For $k \geq 3$,*

$$AR(M_k, 2k) = \max\left\{\binom{k-2}{2} + k^2 - 2, \binom{2k-3}{2} + 3\right\}.$$

Fujito et al conjectured Theorem 1 in [2]. In this paper we prove, for $k \geq 3$,

$$AR(M_k, 2k) \leq \max\{\binom{k-2}{2} + k^2 - 2, \binom{2k-3}{2} + 3\}$$

which completes the proof of Theorem 1.

2 Preliminary results

We begin by establishing properties of colorings with no rainbow matching that use as many colors as possible. Given a coloring h of K_{2k}, we define an auxiliary coloring $\psi_h : E(G) \to \{\text{blue, green, red}\}$, where

$$\psi_h(uv) = \begin{cases} \text{blue} & \text{if } uv \text{ is the only edge with color } h(uv) \\ \text{green} & \text{if } uv \text{ is not the only edge with color } h(uv) \\ & \text{and no edge in } G\backslash\{u,v\} \text{ has color } h(uv) \\ \text{red} & \text{otherwise.} \end{cases}$$

Lemma 1. *Let h be an exact $(AR(M_k, n) - 1)$-edge coloring of K_n such that there does not exist a rainbow k-matching. $\psi_h(e)$ is red or blue for all edges of K_n.*

Proof. Let $uv \in E(K_n)$ such that $\psi_h(uv)$ is green. Let h' be the exact $AR(M_k, n)$-edge coloring of K_n,

$$h'(e) = \begin{cases} AR(M_k, n) & e = uv \\ h(e) & e \neq uv. \end{cases}$$

There must exist a rainbow k-matching. If uv is not in this rainbow k-matching, then h produced a rainbow k-matching. Therefore, uv must be an edge in the k-matching produced by h'. However, since $\psi_h(uv)$ is green every edge colored $h(uv)$ is incident to either u or v. Thus there is no edge in the k-matching with color $h(uv)$. Therefore, the k-matching that is rainbow under h' must have also been rainbow under h, which is a contradiction.

□

For vertex disjoint subgraphs $A, B \subset K_n$, define AB as the set of edges incident to a vertex in A and a vertex in B. (Note: It may be the case that B is a single vertex or an edge.) Define $h(A)$ as the set of colors $\{h(e) \mid e \in E(A)\}$.

Lemma 2. *Let h be an edge coloring of K_6 with no rainbow 3-matching. Let $e = uv \in E(K_6)$ such that $\psi_h(e)$ is blue, and $A = K_6 \backslash \{u, v\}$. The following are true:*

i) $|h(A)| \leq 3$

ii) *If* $|h(A)| = 3$, *then* $|h(Ae) \backslash h(A)| \leq 1$.

iii) *If* $|h(A)| = 2$, *then* $|h(Ae) \backslash h(A)| \leq 2$.

iv) *If* $|h(A)| = 1$, *then* $|h(Ae) \backslash h(A)| \leq 4$.

v) *If* $|h(Ae) \backslash h(A)| \geq 3$, *then* $h(A) = h(Au)$ *or* $h(Av)$.

Proof. *i*) Since e is blue in ψ_h, the edges of every 2-matching of A must be the same color in h. So $1 \leq |h(A)| \leq 3$.

ii) Let $V(A) = \{w, x, y, z\}$. Assume $|h(A)| = 3$ and $|h(Ae) \backslash h(A)| \geq 2$. For some $e_1, e_2 \in Ae$ $h(e_1) \neq h(e_2)$ and $h(e_1), h(e_2) \notin h(A)$. There are three cases. If $\{e_1, e_2\}$ is a 2-matching, say $\{uw, vx\}$, then $\{uw, vx, yz\}$ is a rainbow 3-matching.

If $e_1 = uw$ and $e_2 = ux$, then $h(vy) \in \{h(uw), h(xz)\}$ to avoid $\{uw, vy, xz\}$ being a rainbow 3-matching. Either choice forces $\{ux, vy, wz\}$ to be a rainbow 3-matching.

If $e_1 = uw$ and $e_2 = vw$, then $h(ux) \in \{h(vw), h(yz)\}$ to avoid $\{ux, vw, yz\}$ being a rainbow 3-matching and $h(vy) \in \{h(uw), h(xz)\}$ to avoid $\{vy, uw, xz\}$ being a rainbow 3-matching. Any choice forces $\{ux, vy, wz\}$ to be a rainbow 3-matching. So $|h(Ae)\backslash h(A)| \leq 1$.

iii) This proof is by contradiction using a similar case analysis as *ii*).

iv) Assume $|h(Ae)\backslash h(A)| \geq 5$, then there exists (wlog) edges uw and vx such that $h(uw) \neq h(vx)$ and $h(uw), h(vx) \in (h(Ae)\backslash h(A))$. This implies $\{uw, vx, yz\}$ is a rainbow 3-matching. So $|h(Ae)\backslash h(A)| \leq 4$.

v) Let $|h(Ae)\backslash h(A)| \geq 3$. So $|h(A)| = 1$. If neither $h(Au)$ nor $h(Av)$ is $h(A)$, then there exists (wlog) edges uw and vx such that $h(uw) \neq h(vx)$ and $h(uw), h(vx) \in (h(Ae)\backslash h(A))$. This implies $\{uw, vx, yz\}$ is a rainbow 3-matching. So $h(A) = h(Au)$ or $h(A) = h(Av)$. □

Lemma 3. *$AR(M_3, 6) \leq 7$.*

Proof. Let h be an exact $(AR(M_3, 6)-1)$-edge coloring of K_6 with no rainbow matching. If any edge in ψ_h is blue, then by Lemma 2 $(AR(M_3, 6) - 1) \leq 6$; hence, $AR(M_3, 6) \leq 7$.

If no edge in ψ_h is blue, then by Lemma 1 $\psi_h(e)$ is red for all $e \in E(K_6)$. Since K_6 has 15 edges and each color of h must appear at least two times we get $2(AR(M_3, 6) - 1) \leq 15$ implying $AR(M_3, 6) \leq 8$.

We will show $AR(M_3, 6) \neq 8$. Consider any exact 7-edge coloring, h, with no blue edges and no rainbow 3-matching. Thus six colors are used exactly twice and one color is used exactly three times. Let $uv \in E(K_6)$ such that $h(uv)$ is one of the colors used exactly twice. Then at least two of the 2-matchings in $K_6\backslash\{u, v\}$ have both edges colored the same in h. Let these 2-matchings be $\{wy, xz\}$ and $\{wx, yz\}$. Note that $h(wy) \neq h(wx)$.

The four disjoint 3-matchings $\{uw, vx, yz\}$, $\{uz, vy, wx\}$, $\{uy, vw, xz\}$, and $\{ux, vz, wy\}$ each have an edge with color $h(wy)$ or $h(wx)$. Either $h(wy)$ appears three times, $h(wx)$ appears three times or neither does. In each case, the set of colors $\{h(wy), h(wx)\}$ can color 2 edges of at most one of these matchings. So at least three of these 3-matching will have a single edge with color $h(wy)$ or $h(wx)$ and the other two edges colored the same color but not with color $h(wy)$ or $h(wx)$. Wlog, let $\{uz, vy, wx\}$, $\{uy, vw, xz\}$, and $\{ux, vz, wy\}$ be three such 3-matchings. Then $\{ux, vw, yz\}$ is a rainbow 3-matching. So $AR(M_3, 6) \leq 7$.

□

3 Proof of Main Theorem

In this section we prove, for $k \geq 3$,

$$AR(M_k, 2k) \leq \max\left\{\binom{k-2}{2} + k^2 - 2, \binom{2k-3}{2} + 3\right\}$$

which completes the proof of the Main Theorem. The proof is inductive with Lemma 3 as the base case. Let $k > 3$ and

$$f(k) = \begin{cases} \binom{k-2}{2} + k^2 - 2 & \text{if } 4 \leq k \leq 6 \\ \\ \binom{2k-3}{2} + 3 & \text{if } 7 \leq k. \end{cases}$$

We will show that if $AR(M_{k-1}, 2(k-1)) \leq f(k-1)$, then $AR(M_k, 2k) \leq f(k)$.

Throughout this section, the following assumptions will be made. Let r be the maximum integer such that there exists an exact r-edge coloring of K_{2k} with no rainbow k-matching (i.e, $r = AR(M_k, 2k) - 1$). We assume $r \geq f(k)$ and $AR(M_{k-1}, 2(k-1)) \leq f(k-1)$. We will show, by contradiction, that $r < f(k)$.

Let h be an exact r-edge coloring. For each $v \in V(K_{2k})$, define $d_R(v)$ (and $d_B(v)$) as the number of edges incident to v and colored red (blue) by ψ_h.

Lemma 4. *If there exists an edge $e = uv \in E(K_{2k})$ such that $\psi_h(e)$ is blue and*

$$|(h(Ae) \cup h(e)) \backslash h(A)| \leq \begin{cases} 3k - 4 & \textit{if } 4 \leq k \leq 5 \\ 4k - 10 & \textit{if } 6 \leq k, \end{cases}$$

where $A = K_{2k} \backslash \{u, v\}$, then there is a rainbow k-matching.

Proof. Observe that $r = |h(A)| + |(h(Ae) \cup h(e)) \backslash h(A)|$. If $4 \leq k \leq 5$, then $f(k) \leq |h(A)| + 3k - 4$ and $f(k-1) \leq f(k) - (3k - 4)$. Therefore, $f(k-1) \leq |h(A)|$, so there exists a rainbow $(k-1)$-matching M in A. If $k \geq 6$ then $f(k) \leq |h(A)| + 4k - 10$ and $f(k-1) \leq f(k) - (4k - 10)$. Therefore, $f(k-1) \leq |h(A)|$, so there exists a rainbow $(k-1)$-matching M in A. In either case $\psi_h(e)$ is blue, so $M \cup e$ is a rainbow k-matching.

□

Lemma 5. *If h contains no rainbow k-matching, then for any $v \in V(K_{2k})$, $d_R(v) \leq \min(k, 6)$ or $d_R(v) = 2k - 1$.*

Proof. Assume there exist a vertex v with $\min(k, 6) < d_R(v) < 2k-1$. There exists some vertex u such that $\psi_h(uv)$ is blue. The number of edges incident to u or v is $4k-3 = d_B(u)+d_B(v)+d_R(u)+d_R(v)-1$. Let $A = K_{2k}\backslash\{u, v\}$. Note that if $h(e) \notin h(A)$ and $\psi_h(e)$ is red, then by the definition of ψ_h, $h(e)$ must appear on both an edge incident to u and an edge incident to v. Then

$$\begin{aligned} |(h(Ae) \cup h(e))\backslash h(A)| &\leq d_B(u) + d_B(v) - 1 + \min(d_R(u), d_R(v)) \\ &\leq 4k - 3 - \max(d_R(u), d_R(v)). \end{aligned}$$

If $k \geq 6$, then $\max(d_R(u), d_R(v)) \geq 7$ which implies $|(h(Ae)\cup h(e))\backslash h(A)| \leq 4k - 10$ and by Lemma 4 there exists a rainbow k-matching. For $k = 5$, $|(h(Ae)\cup h(e))\backslash h(A)| \leq 4k-9 = 3k-4$ and for $k = 4$, $|(h(Ae)\cup h(e))\backslash h(A)| \leq 4k-8 = 3k-4$. Therefore, both of these cases result in a rainbow k-matching by Lemma 4.

□

Lemma 6. *Let M be the largest matching such that for all $e \in M$, $\psi_h(e)$ is blue. If h has no rainbow k-matching, then the following are true:*

i) $|M| = k - 2$ *or* $k - 3$,

ii) There is at most one vertex $v \notin V(M)$ *with* $d_R(v) < 2k - 1$, *and*

iii) For all $v \in V(M)$, $d_R(v) \geq 2k - 2|M| - 1$.

Proof. The proof begins by showing that $|M| > 0$. Let $\mathcal{B} = \{e \in E(K_{2k})|\ \psi_h(e)$ is blue$\}$. Each edge of the K_{2k} counts each color represented by a blue edge once and each color represented by a red edge at least twice. Therefore, $\binom{2k}{2} + |\mathcal{B}| \geq 2f(k)$. For $k > 3$, $2f(k) - \binom{2k}{2} > 0$. Since $\mathcal{B}$ is nonempty, M is nonempty.

Let $A = K_{2k} - V(M)$. $|V(A)|$ must be an even integer greater than 2. $|V(A)| = 2$ implies $M \cup A$ is a rainbow k-matching. Since M is maximal, $\psi_h(e)$ is red for all $e \in E(A)$.

If $|V(A)| \geq 8$ and there is an edge $uv \in AM$ such that $u \in V(A)$ and $\psi_h(uv)$ is blue, then $7 \leq d_R(u) < 2k - 1$. On the other hand, if all the edges of AM are red, then $8 \leq d_R(v) < 2k - 1$ for all $v \in V(M)$. In either case, Lemma 5 implies the existence of a rainbow $k-$matching. So $|V(A)|$ is 4 or 6; hence, $|M| = k-2$ or $k-3$.

Let $|M| = k-2$. Assume $v_1, v_2 \in V(A)$ with $d_R(v_1), d_R(v_2) < 2k-1$. If $xy \in M$ and $\psi_h(v_1x)$ is blue, then $\psi_h(v_2y)$ must be red, otherwise $(M\backslash\{xy\}) \cup \{v_1x, v_2y\}$ is a larger blue matching. So each blue edge of Mv_1 forces a unique red edge of Mv_2. Since ψ_h is red for all edges of A, by Lemma 5, the number of edges in Mv_i that are red is at most $k-3$. This implies there exist at least $2(k-2)-(k-3) = k-1$ blue edges in Mv_i. This is a contradiction since there are at least $k-1$ blue edges in Mv_1 and at most $k-3$ red edges in Mv_2 and there must be at least as many red edges as blue edges.

When $|M| = k-3$, a similar contradiction will be obtained since there is at most 1 red edge in Mv_i and at least 3 blue edges in Mv_i.

By $ii)$, there exists at least $2k-2|M|-1$ vertices that are incident to all red edges in ψ_h; hence, $iii)$ is true. □

Let $M = \{e_1, e_2, \ldots, e_m\}$ be a maximum blue matching. Let $A = K_{2k}\backslash B$, where B is the complete subgraph of K_{2k} induced by M. By Lemma 6, $|A| = 4$ or 6. Let $a = |h(A)|$, $b = |h(B)\backslash h(A)|$, and $c_i = |h(Ae_i)\backslash(h(A) \cup h(B) \cup \bigcup_{j=1}^{i-1} h(Ae_j))|$ for $1 \le i \le m$. So $r = a + b + c_1 + c_2 + \ldots + c_m$.

Case I. $m = k-2$:

If h has no rainbow k-matching and $m = k-2$ then for each i, the subgraph induced by $A \cup e_i$ doesn't contain a rainbow 3-matching.

If $a = 3$, then $c_i \le 1$ for all i by Lemma 2. Since $b \le \binom{2k-4}{2}$, $r \le 3 + \binom{2k-4}{2} + k - 2$. Thus in this case $r \le \binom{2k-4}{2} + k + 1 < f(k)$ for all $k \ge 4$.

If $a = 2$, then $c_i \le 2$ for all i by Lemma 2. Since $b \le \binom{2k-4}{2}$, $r \le 2 + \binom{2k-4}{2} + 2(k-2) = \binom{2k-3}{2} + 2$. So $r < f(k)$ for all $k \ge 4$.

If $a = 1$, then $c_i \le 4$ for all i by Lemma 2. We know that $r \ge f(k)$, so $r \ge \binom{2k-3}{2} + 3$. Since $b \le \binom{2k-4}{2}$, $\sum_{i=1}^{k-2} c_i \ge r - 1 - \binom{2k-4}{2}$, which implies $\sum_{i=1}^{k-2} c_i \ge (\binom{2k-3}{2} + 3) - 1 - \binom{2k-4}{2} = 2k-2$. Therefore, there exists an integer α such that $c_\alpha \ge \frac{2k-2}{k-2} > 2$. Let $e_\alpha = xy$ and $C = K_{2k}\backslash\{x, y\}$.

There are $4k-3$ edges incident to x or y so $|(h(Ce) \cup h(e))\backslash h(C)| \le 4k-3$. Without loss of generality, $h(A) = h(Ax)$ by Lemma 2, so at least 4 edges incident to x are assigned a color by h which also appears in C.

By Lemma 6, y is incident to at least 3 edges colored red by ψ_h, in particular, at least 3 red edges in Ay. Say p_1, p_2, and p_3 are such edges. If the color $h(p_1)$ appears in $h(Ce_\alpha) \cup h(e_\alpha)\backslash h(C)$, then there must be some edge in Cx with color $h(p_1)$. So there are two edges in Ce_α colored $h(p_1)$. Similarly, there are two edges in Ce_α colored each of $h(p_2)$ and $h(p_3)$. So $|(h(Ce_\alpha) \cup h(e_\alpha)) \backslash h(C)| \leq (4k-3)-4-3 = 4k-10$. Note $3k-4 \geq 4k-10$ when $k \leq 6$ so for all k this is a contradiction by Lemma 4.

Case II. $m = k-3$:
If h has no rainbow k-matching and $m = k-3$ then for each $v \in V(B)$, $5 \leq d_R(v)$ by Lemma 6. Every vertex $v \in B$ has at least one blue edge, so by Lemma 5, $d_R(v) \leq 6$ and $5 \leq k$. Thus, there exist two edges $p_1 = vx_1, p_2 = vx_2 \in E(B)$ such that $\psi_h(p_i)$ is blue.

We next show that $|h((K_{2k} \backslash \{v, x_i\})p_i) \cup (h(p_i) \backslash h(K_{2k} \backslash \{v, x_i\}))| \leq 4k-10$ for $i = 1$ or 2. That is to say, that p_1 or p_2 will satisfy the conditions of Lemma 4 resulting in a rainbow $k-$matching.

Let $H_i = h((K_{2k} \backslash \{v, x_i\})p_i) \cup (h(p_i) \backslash h(K_{2k} \backslash \{v, x_i\}))$ and $R_v = \{vw | \psi_h(vw)$ is red$\}$. If $vw \in R_v$ and $h(vw) \in H_i$, then there exists an edge incident to x_i that has the color $h(vw)$; therefore, $h(vw)$ is in at most one of H_1 or H_2. This implies that $|h(R_v) \cap H_i| \leq 3$ for $i = 1$ or 2, since $|R_v| \leq 6$. Without loss of generality, say $|h(R_v) \cap H_1| \leq 3$. There are at most $4k-13$ edges incident to v or x_1 that are colored blue by ψ_h. Therefore, $|H_1| \leq (4k-13)+3 = 4k-10$. By Lemma 4, we obtain a contradiction.

Therefore, $r < f(k)$ so $AR(M_k, 2k) - 1 < f(k)$, proving our result.

References

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