Propagation time for zero forcing on a graph

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Abstract

Zero forcing (also called graph infection) on a simple, undirected graph G is based on the color-change rule: If each vertex of G is colored either white or black, and vertex v is a black vertex with only one white neighbor w, then change the color of w to black. A minimum zero forcing set is a set of black vertices of minimum cardinality that can color the entire graph black using the color change rule. The propagation time of a zero forcing set B of graph G is the minimum amount of time that it takes to force all the vertices of G black, starting with the vertices in B black and performing independent forces simultaneously. The minimum and maximum propagation times of a graph are taken over all minimum zero forcing sets of the graph. It is shown that a connected graph of order at least two has more than one minimum zero forcing set realizing minimum propagation time. Graphs G having extreme minimum propagation times $|G|-1$, $|G|-2$, and 0 are characterized, and results regarding graphs having minimum propagation time 1 are established. It is shown that the diameter is an upper bound for maximum propagation time for a tree, but in general propagation time and diameter of a graph are not comparable.

Keywords zero forcing number, propagation time, graph

AMS subject classification 05C50

1 Propagation time

All graphs are simple, finite, and undirected. In a graph G where some vertices are colored black and the remaining vertices are colored white, the *color change rule* is: If v is black and w is the only white neighbor of v, then change the color of w to black; if we apply the color change rule to v to change the color of w, we say v *forces* w and write $v \to w$ (note that there may be a choice involved, since only one vertex actually forces w, but more than one may be able to). Given an initial set B of black vertices, the *final coloring* of B is the set of black vertices that results from applying the color change rule until no more changes are possible. The final coloring is unique [1]. A *zero forcing set* is an initial set B of

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vertices such that the final coloring of B is $V(G)$. A *minimum zero forcing set* of a graph G is a zero forcing set of G of minimum cardinality, and the *zero forcing number*, denoted $\mathrm{Z}(G)$, is the cardinality of a minimum zero forcing set.

Zero forcing, also known as graph infection or graph propagation, was introduced independently in [1] for study of minimum rank problems in combinatorial matrix theory, and in [3] for study of control of quantum systems. Propagation time of a zero forcing set, which describes the amount of time needed to fully color a graph performing independent forces simultaneously, was implicit in [3] and explicit in [7]. In this paper we systematically study propagation time.

Definition 1.1. Let $G = (V, E)$ be a graph and B a zero forcing set of G. Define $B^{(0)} = B$, and for $t \geq 0$, $B^{(t+1)}$ is the set of vertices w for which there exists a vertex $b \in \bigcup_{s=0}^{t} B^{(s)}$ such that w is the only neighbor of b not in $\bigcup_{s=0}^{t} B^{(s)}$. The *propagation time of* B in G, denoted $\mathrm{pt}(G, B)$, is the smallest integer t_0 such that $V = \bigcup_{t=0}^{t_0} B^{(t)}$.

Two minimum zero forcing sets of the same graph may have different propagation times, as the following example illustrates.

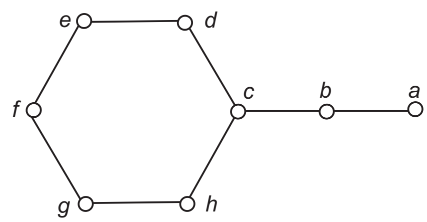

Figure 1: The graph G for Example 1.2

Example 1.2. Let G be the graph in Figure 1. Let $B_1 = \{g, h\}$ and $B_2 = \{a, d\}$. Then ${B_1}^{(1)} = \{f, c\}$, ${B_1}^{(2)} = \{e\}$, ${B_1}^{(3)} = \{d\}$, ${B_1}^{(4)} = \{b\}$, and ${B_1}^{(5)} = \{a\}$, so $\mathrm{pt}(G, B_1) = 5$. However, ${B_2}^{(1)} = \{b\}$, ${B_2}^{(2)} = \{c\}$, ${B_2}^{(3)} = \{e, h\}$, and ${B_2}^{(4)} = \{f, g\}$, so $\mathrm{pt}(G, B_2) = 4$.

Definition 1.3. The *minimum propagation time* of G is

$$\mathrm{pt}(G) = \min\{\mathrm{pt}(G, B) \mid B \text{ is a minimum zero forcing set of } G\}.$$

Definition 1.4. Two minimum zero forcing sets B_1 and B_2 of a graph G are *isomorphic* if there is a graph automorphism φ of G such that $\varphi(B_1) = B_2$.

It is obvious that isomorphic zero forcing sets have the same propagation time, but a graph may have non-isomorphic minimum zero forcing sets and have the property that all minimum zero forcing sets have the same propagation time.

Example 1.5. Up to isomorphism, the minimum zero forcing sets of the dart shown in Figure 2 are $\{a, c\}$, $\{b, c\}$, and $\{c, d\}$. Each of these sets has propagation time 3.

The minimum propagation time of a graph G is not subgraph monotone. For example, it is easy to see that the 4-cycle has $\mathrm{Z}(C_4) = 2$ and $\mathrm{pt}(C_4) = 1$. By deleting one edge of C_4, it becomes a path P_3, which has $\mathrm{Z}(P_3) = 1$ and $\mathrm{pt}(P_3) = 2$. Furthermore, since minimum propagation time is not subgraph monotone, it is not minor monotone.

A minimum zero forcing set that achieves minimum propagation time plays a central role and we name such a set.

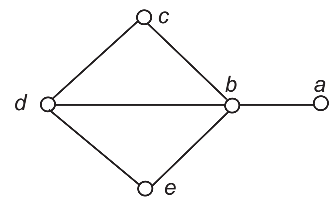

Figure 2: The dart

Definition 1.6. A subset B of vertices of G is an *efficient zero forcing set* for G if B is a minimum zero forcing set of G and $\mathrm{pt}(G,B) = \mathrm{pt}(G)$. Define

$$\mathrm{Eff}(G) = \{B \mid B \text{ is an efficient zero forcing set of } G\}.$$

We can also consider maximum propagation time.

Definition 1.7. The *maximum propagation time* of G is defined as

$$\mathrm{PT}(G) = \max\{\mathrm{pt}(G,B) \mid B \text{ is a minimum zero forcing set of } G\}.$$

The following bounds are obvious.

Observation 1.8. *Let G be a graph. Then*

$$\begin{aligned} \frac{|G| - \mathrm{Z}(G)}{\mathrm{Z}(G)} &\leq \mathrm{pt}(G); \\ \mathrm{PT}(G) &\leq |G| - \mathrm{Z}(G). \end{aligned}$$

Definition 1.9. The *propagation time interval* of G is defined as

$$[\mathrm{pt}(G), \mathrm{PT}(G)] = \{\mathrm{pt}(G), \mathrm{pt}(G)+1, \ldots, \mathrm{PT}(G)-1, \mathrm{PT}(G)\}.$$

The *propagation time discrepancy* of G is defined as

$$\mathrm{pd}(G) = \mathrm{PT}(G) - \mathrm{pt}(G).$$

It is not the case that every integer in the propagation time interval is the propagation time of a minimum zero forcing set; this can be seen in the next example. Let $S(e_1, e_2, e_3)$ be the generalized star with three arms having e_1, e_2, e_3 vertices with $e_1 \leq e_2 \leq e_3$; $S(2, 5, 11)$ is shown in Figure 3.

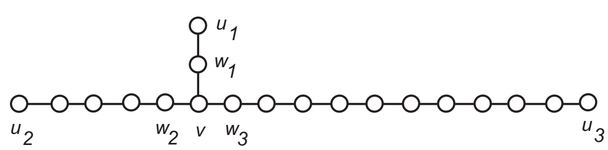

Figure 3: The tree $S(2, 5, 11)$

Example 1.10. Consider $S(e_1, e_2, e_3)$ with $1 < e_1 < e_2 < e_3$. The vertices of degree one are denoted by u_1, u_2, u_3, the vertex of degree three is denoted by v, and neighbors of v are denoted by w_1, w_2, w_3. The minimum zero forcing sets and their propagation times are shown in Table 1. Observe that the propagation time interval of $S(2, 5, 11)$ is $[12, 16]$, but there is no minimum zero forcing set with propagation time 14. The propagation discrepancy is $\mathrm{pd}(S(2, 5, 11)) = 4$.

Table 1: Minimum zero forcing sets and propagation times of $S(e_1, e_2, e_3)$

B	$\text{pt}(S(e_1, e_2, e_3), B)$	$\text{pt}(S(2, 5, 11), B)$
$\{u_1, u_2\}$	$e_2 + e_3 - 1$	15
$\{u_3, w_2\}$	$e_2 + e_3 - 1$	15
$\{u_3, w_1\}$	$e_2 + e_3$	16
$\{u_1, u_3\}$	$e_2 + e_3 - 1$	15
$\{u_2, w_3\}$	$e_2 + e_3 - 1$	15
$\{u_2, w_1\}$	$e_2 + e_3$	16
$\{u_2, u_3\}$	$e_1 + e_3 - 1$	12
$\{u_1, w_3\}$	$e_1 + e_3 - 1$	12
$\{u_1, w_2\}$	$e_1 + e_3$	13

The next remark provides a necessary condition for a graph G to have $\text{pd}(G) = 0$.

Remark 1.11. Let G be a graph. Then every minimum zero forcing set of G is an efficient zero forcing set if and only if $\text{pd}(G) = 0$. In [2], it is proven that the intersection of all minimum zero forcing sets is the empty set. Hence, $\text{pd}(G) = 0$ implies $\bigcap_{B \in \text{Eff}(G)} B = \emptyset$.

In Section 2 we establish properties of efficient zero forcing sets, including that no connected graph of order more than one has a unique efficient zero forcing set. In Section 3 we characterize graphs having extreme propagation time. In Section 4 we examine the relationship between propagation time and diameter.

2 Efficient zero forcing sets

In [2] it was shown that for a connected graph of order at least two, there must be more than one minimum zero forcing set and furthermore, no vertex is in every minimum zero forcing set. This raises the questions of whether the analogous properties are true for efficient zero forcing sets (Questions 2.1 and 2.14 below).

Question 2.1. *Is there a connected graph of order at least two that has a unique efficient zero forcing set?*

We show that the answer to Question 2.1 is negative. First we need some terminology. For a given zero forcing set B of G, construct the final coloring, listing the forces in the order in which they were performed. This list is a *chronological list of forces* of B [5]. Many definitions and results concerning lists of forces that have appeared in the literature involve chronological (ordered) lists of forces. For the study of propagation time, the order of forces is often dictated by performing a force as soon as possible (propagating). Thus unordered sets of forces are more useful than ordered lists when studying propagation time, and we extend terminology from chronological lists of forces to sets of forces.

Definition 2.2. Let $G = (V, E)$ be a graph, B a zero forcing set of G. The unordered set of forces in a chronological list of forces of B is called *a set of forces* of B.

Observe that if B is a zero forcing set and $\mathcal{F}$ is a set of forces of B, then the cardinality of $\mathcal{F}$ is $|G| - |B|$. The ideas of terminus and reverse set of forces, introduced in [2] for a chronological list of forces and defined below for a set of forces, are used to answer Question 2.1 negatively (by constructing the terminus of a set of forces of an efficient zero forcing set).

Definition 2.3. Let G be a graph, let B be a zero forcing set of G, and let $\mathcal{F}$ be a set of forces of B. The *terminus* of $\mathcal{F}$, denoted $\mathrm{Term}(\mathcal{F})$, is the set of vertices that do not perform a force in $\mathcal{F}$. The *reverse set of forces* of $\mathcal{F}$, denoted here as $\mathrm{Rev}(\mathcal{F})$, is the result of reversing each force in $\mathcal{F}$. A *forcing chain* of $\mathcal{F}$ is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that for $i = 1, \ldots, k-1$, v_i forces v_{i+1} in $\mathcal{F}$.

The name "terminus" reflects the fact that a vertex does not perform a force in $\mathcal{F}$ if and only if it is the end point of a forcing chain (the latter is the definition used in [2], where such a set is called a reversal of B). In [2], it is shown that if B is a zero forcing set of G and $\mathcal{F}$ is a chronological list of forces, then the terminus of $\mathcal{F}$ is also a zero forcing set of G, with the reverse chronological list of forces (to construct a reverse chronological list of forces of $\mathcal{F}$, write the chronological list of forces in reverse order and reverse each force in $\mathcal{F}$).

Observation 2.4. *Let G be a graph, B a minimum zero forcing set of G, and $\mathcal{F}$ a set of forces of B. Then* $\mathrm{Rev}(\mathcal{F})$ *is a set of forces of* $\mathrm{Term}(\mathcal{F})$ *and* $B = \mathrm{Term}(\mathrm{Rev}(\mathcal{F}))$.

When studying propagation, it is natural to examine sets of forces that achieve minimum propagation time.

Definition 2.5. Let $G = (V, E)$ be a graph and B a zero forcing set of G. For a set of forces $\mathcal{F}$ of B, define $\mathcal{F}^{(0)} = B$ and for $t \geq 0$, $\mathcal{F}^{(t+1)}$ is the set of vertices w such that the force $v \to w$ appears in $\mathcal{F}$, $w \notin \bigcup_{i=0}^{t} \mathcal{F}^{(i)}$, and w is the only neighbor of v not in $\bigcup_{i=0}^{t} \mathcal{F}^{(i)}$. The *propagation time of* $\mathcal{F}$ in G, denoted $\mathrm{pt}(G, \mathcal{F})$, is the least t_0 such that $V = \bigcup_{t=0}^{t_0} \mathcal{F}^{(t)}$.

Let $G = (V, E)$ be a graph, let B be a zero forcing set of G, and let $\mathcal{F}$ be a set of forces of B. Clearly, $\bigcup_{i=0}^{t} \mathcal{F}^{(i)} \subseteq \bigcup_{i=0}^{t} B^{(i)}$ for all $t = 0, \ldots, \mathrm{pt}(G, B)$, and $\mathcal{F}$ is a propagating set of forces if and only if $\mathcal{F}^{(t)} = B^{(t)}$, for $t = 0, \ldots, \mathrm{pt}(G, B)$.

Definition 2.6. Let $G = (V, E)$ be a graph and let B be a zero forcing set of G. A set of forces $\mathcal{F}$ is *efficient* if $\mathrm{pt}(G, \mathcal{F}) = \mathrm{pt}(G)$. Define

$$\mathcal{F}_{eff}(G) = \{\mathcal{F} \,|\, \mathcal{F} \text{ is an efficient set of forces of a minimum zero forcing set } B \text{ of } G\}.$$

If $\mathcal{F}$ is an efficient set of forces of a minimum zero forcing set B of G, then B is necessarily an efficient zero forcing set. However, not every efficient set of forces conforms to the propagation process.

Example 2.7. Let G be the graph in Figure 4. Since every degree one vertex must be an endpoint of a forcing chain and since $B = \{x, z, v\}$ is a zero forcing set, $\mathrm{Z}(G) = 3$. Since (v, c, d, e, f, w) or (w, f, e, d, c, v) must be a forcing chain for any set of forces of a minimum zero forcing set, $\mathrm{pt}(G) = 3$. Then B is an efficient zero forcing set with efficient set of forces $\mathcal{F} = \{v \to c, z \to a, c \to d, a \to b, d \to e, b \to y, e \to f, f \to w\}$. Observe that $b \in \mathcal{F}^{(2)}$ (i.e., b does not turn black until time $t = 2$ in $\mathcal{F}$), but $b \in B^{(1)}$ (b can be forced by x at time $t = 1$).

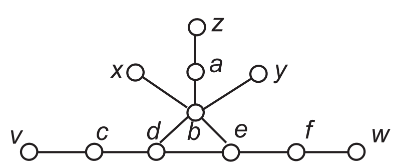

Figure 4: The graph G for Example 2.7

Definition 2.8. Let $G = (V, E)$ be a graph, B a zero forcing set of G, and $\mathcal{F}$ a set of forces of B. Define $Q_0(\mathcal{F}) = \mathrm{Term}(\mathcal{F})$ and for $t = 1, \ldots, \mathrm{pt}(G, \mathcal{F})$, define

$$Q_t(\mathcal{F}) = \{v \in V \,|\, \exists w \ \in \mathcal{F}^{(\mathrm{pt}(G,\mathcal{F})-t+1)} \text{ such that } v \to w\}.$$

Observe that $V = \bigcup_{t=0}^{\mathrm{pt}(G,\mathcal{F})} Q_t(\mathcal{F})$.

Lemma 2.9. *Let $G = (V, E)$ be a graph, B a zero forcing set of G, and $\mathcal{F}$ a set of forces of B. Then $Q_t(\mathcal{F}) \subseteq \bigcup_{i=0}^{t} \mathrm{Rev}(\mathcal{F})^{(i)}$.*

Proof. Recall that $\mathrm{Rev}(\mathcal{F})$ is a set of forces of $\mathrm{Term}(\mathcal{F})$. The result is established by induction on t. Initially, $Q_0(\mathcal{F}) = \mathrm{Term}(\mathcal{F}) = \mathrm{Rev}(\mathcal{F})^{(0)}$. Assume that for $0 \le s \le t$, $Q_s(\mathcal{F}) \subseteq \bigcup_{i=0}^{s} \mathrm{Rev}(\mathcal{F})^{(i)}$. Let $v \in Q_{t+1}(\mathcal{F})$. In $\mathcal{F}$, $v \to u$ at time $\mathrm{pt}(G, \mathcal{F}) - t$. In $\mathcal{F}$, u cannot perform a force until time $\mathrm{pt}(G, B) - t + 1$ or later, so $u \in \bigcup_{i=0}^{t} Q_i(\mathcal{F}) \subseteq \bigcup_{i=0}^{t} \mathrm{Rev}(\mathcal{F})^{(i)}$. If $x \in N(u) \setminus \{v\}$ then in $\mathcal{F}$ x cannot perform a force before time $\mathrm{pt}(G, \mathcal{F}) - t + 1$, so $x \in \bigcup_{i=0}^{t} Q_i(\mathcal{F}) \subseteq \bigcup_{i=0}^{t} \mathrm{Rev}(\mathcal{F})^{(i)}$. So if $v \notin \bigcup_{i=0}^{t} \mathrm{Rev}(\mathcal{F})^{(i)}$, then $v \in \mathrm{Rev}(\mathcal{F})^{(t+1)}$. Thus $v \in \bigcup_{i=0}^{t+1} \mathrm{Rev}(\mathcal{F})^{(i)}$. □

Corollary 2.10. *Let $G = (V, E)$ be a graph, B a minimum zero forcing set of G, and $\mathcal{F}$ a set of forces of B. Then*

$$\mathrm{pt}(G, \mathrm{Rev}(\mathcal{F})) \le \mathrm{pt}(G, \mathcal{F}).$$

The next result follows from Corollary 2.10 and Observation 2.4.

Theorem 2.11. *Let $G = (V, E)$ be a graph, B an efficient zero forcing set of G, and $\mathcal{F}$ an efficient set of forces of B. Then* $\mathrm{Rev}(\mathcal{F})$ *is an efficient set of forces and* $\mathrm{Term}(\mathcal{F})$ *is an efficient zero forcing set. Every efficient zero forcing set is the terminus of an efficient set of forces of an efficient zero forcing set.*

The next result answers Question 2.1 negatively.

Theorem 2.12. *Let G be a connected graph of order greater than one. Then $|\,\mathrm{Eff}(G)| \ge 2$.*

Proof. Let $B \in \mathrm{Eff}(G)$ and let $\mathcal{F}$ be an efficient set of forces of B. By Theorem 2.11, $\mathrm{Term}(B) \in \mathrm{Eff}(G)$. Since G is a connected graph of order greater than one, $B \ne \mathrm{Term}(\mathcal{F})$. □

We now consider the intersection of efficient zero forcing sets. The next result is immediate from Theorem 2.11.

Corollary 2.13. *Let G be a graph. Then* $\displaystyle\bigcap_{B \in \mathrm{Eff}(G)} B = \bigcap_{\mathcal{F} \in \mathcal{F}_{eff}(G)} \mathrm{Term}(\mathcal{F}).$

Question 2.14. *Is there a connected graph G of order at least two and a vertex $v \in V(G)$ such that v is in every efficient zero forcing set?*

The next example provides an affirmative answer.

Example 2.15. The wheel W_5 is the graph shown in Figure 5. The efficient zero forcing set $\{a, b, c\}$

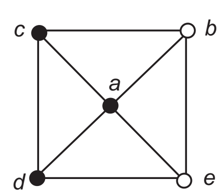

Figure 5: The wheel W_5

of W_5 shows that $\mathrm{pt}(W_5) = 1$. Up to isomorphism, there are two types of minimum zero forcing sets in W_5. One set contains a and two other vertices that are adjacent to each other; the other contains three vertices other than a. The latter is not an efficient zero forcing set of W_5, because its propagation time is 2. The possible choices for an efficient zero forcing set are $\{a, b, c\}, \{a, c, d\}, \{a, d, e\}$, or $\{a, b, e\}$. Therefore, $\bigcap_{B \in \mathrm{Eff}(G)} B = \{a\}$.

We examine the effect of a nonforcing vertex in an efficient zero forcing set. This result will be used in Section 3.2.

Theorem 2.16. *For a vertex v of a graph G, there exists an efficient zero forcing set B containing v and an efficient set of forces $\mathcal{F}$ in which v does not perform a force if and only if $\mathrm{pt}(G - v) = \mathrm{pt}(G)$ and $\mathrm{Z}(G - v) = \mathrm{Z}(G) - 1$.*

Proof. It was shown in [4] that $\mathrm{Z}(G) = \mathrm{Z}(G - v) + 1$ if and only if there exists a minimum zero forcing set B containing v and set of forces $\mathcal{F}$ in which v does not perform a force.

Assume there exists an efficient zero forcing set B containing v and an efficient set of forces $\mathcal{F}$ in which v does not perform a force, so $\mathrm{Z}(G - v) = \mathrm{Z}(G) - 1$. Then $B' = B \setminus \{v\}$ is a minimum zero forcing set of $G - v$ with a set of forces $\mathcal{F}$. Thus $\mathrm{pt}(G - v) \le \mathrm{pt}(G - v, \mathcal{F}) = \mathrm{pt}(G, \mathcal{F}) = \mathrm{pt}(G)$. Let B' be an efficient zero forcing set of $G - v$ and $\mathcal{F}'$ an efficient set of forces of B'. Then $B = B' \cup \{v\}$ is a minimum zero forcing set of G with a set of forces $\mathcal{F}'$. Thus $\mathrm{pt}(G) \le \mathrm{pt}(G, \mathcal{F}') = \mathrm{pt}(G - v, \mathcal{F}') = \mathrm{pt}(G - v)$.

For the converse, assume $\mathrm{pt}(G - v) = \mathrm{pt}(G)$ and $\mathrm{Z}(G - v) = \mathrm{Z}(G) - 1$. Let B' be a efficient zero forcing set of $G - v$ and $\mathcal{F}'$ an efficient set of forces of B'. Then $B = B' \cup \{v\}$ is a minimum zero forcing set of G with the set of forces $\mathcal{F}'$. Since $\mathrm{pt}(G - v) = \mathrm{pt}(G)$, B is an efficient zero forcing set of G and $\mathcal{F}'$ is an efficient set of forces in which v does not perform a force. □

3 Graphs with extreme minimum propagation time

For any graph G, it is clear that $0 \le \mathrm{pt}(G) \le \mathrm{PT}(G) \le |G| - 1$. In this section we consider the extreme values $|G| - 1$, $|G| - 2$, i.e., high propagation time, and, 0 and 1, i.e., low propagation time.

3.1 High propagation time

The case of propagation time $|G| - 1$ is easy.

Proposition 3.1. *For a graph G, the following are equivalent.*

1. $\mathrm{pt}(G) = |G| - 1$.
2. $\mathrm{PT}(G) = |G| - 1$.
3. $\mathrm{Z}(G) = 1$.
4. *G is a path.*

Proof. It is well known that $\mathrm{Z}(G) = 1$ if and only if G is a path, and in this case clearly $\mathrm{pt}(G) = |G| - 1 = \mathrm{PT}(G)$. Clearly $\mathrm{pt}(G) = |G| - 1$ implies $\mathrm{PT}(G) = |G| - 1$. Suppose $\mathrm{PT}(G) = |G| - 1$. Since $\mathrm{PT}(G) \le |G| - \mathrm{Z}(G)$, $|G| - 1 \le |G| - \mathrm{Z}(G)$. This implies that $\mathrm{Z}(G) = 1$ and $\mathrm{PT}(G) = |G| - 1$. □

We now consider graphs that have maximum or minimum propagation time equal to order minus two.

Observation 3.2. *For a graph G,*

1. $\mathrm{pt}(G) = |G| - 2$ *implies* $\mathrm{PT}(G) = |G| - 2$*, but not conversely (see Lemma 3.4 for an example).*

2. $\mathrm{pt}(G) = |G| - 2$ *if and only if* $\mathrm{Z}(G) = 2$ *and exactly one force is performed at each time for every minimum zero forcing set.*

3. $\mathrm{PT}(G) = |G| - 2$ *if and only if* $\mathrm{Z}(G) = 2$ *and there exists a minimum zero forcing set such that exactly one force performed at each time.*

Lemma 3.3. *Let G be a disconnected graph. Then the following are equivalent.*

1. $\mathrm{pt}(G) = |G| - 2$.

2. $\mathrm{PT}(G) = |G| - 2$.

3. $G = P_{n-1}\dot{\cup}P_1$.

Proof. Clearly $G = P_{n-1}\dot{\cup}P_1 \Rightarrow \mathrm{pt}(G) = |G|-2 \Rightarrow \mathrm{PT}(G) = |G|-2$. So assume $\mathrm{PT}(G) = |G|-2$. Since $\mathrm{Z}(G) = 2$, G has exactly two components. At least one component of G is an isolated vertex (otherwise, more than one force occurs at time step one), and so $G = P_{n-1}\dot{\cup}P_1$. □

A graph G is a *graph on two parallel paths* if $V(G)$ can be partitioned into disjoint subsets U_1 and U_2 so that the induced subgraphs $P_i = G[U_i], i = 1, 2$ are paths, G can be drawn in the plane with the paths P_1 and P_2 as parallel line segments, and edges between the two paths (drawn as line segments, not curves) do not cross; such a drawing is called a *standard drawing*. The paths P_1 and P_2 are called the *parallel paths* (for this representation of G as a graph on two parallel paths).

Let G be a graph on two parallel paths P_1 and P_2. If $v \in V(G)$, then $\mathrm{path}(v)$ denotes the one of the parallel paths that contains v and $\overline{\mathrm{path}}(v)$ denotes the other of the parallel paths. Fix an ordering of the vertices in each of P_1 and P_2 that is consistent for both paths in a standard drawing. With this ordering, let $\mathrm{first}(P_i)$ and $\mathrm{last}(P_i)$ denote the first and last vertices of $P_i, i = 1, 2$. If $v, w \in V(P_i)$, then $v \prec w$ means v precedes w in the order on P_i. Furthermore, if $v \in V(P_i)$ and $v \neq \mathrm{last}(P_i)$, $\mathrm{next}(v)$ is the neighbor of v in P_i such that $v \prec \mathrm{next}(v)$; $\mathrm{prev}(v)$ is defined analogously (for $v \neq \mathrm{first}(P_i)$).

Row [6] has shown that $Z(G) = 2$ if and only if G is a graph on two parallel paths. Observe that for any graph having $\mathrm{Z}(G) = 2$, a set of forces $\mathcal{F}$ of a minimum zero forcing set naturally produces a representation of G as a graph on two parallel paths with the parallel paths being the forcing chains. The ordering of the vertices in the parallel paths is the forcing order.

Lemma 3.4. *For a tree G,* $\mathrm{PT}(G) = |G| - 2$ *if and only if* $G = S(1, 1, n - 3)$ *(sometimes called a T-shaped tree). The graph $K_{1,3}$ is the only tree for which* $\mathrm{pt}(G) = |G| - 2$.

Proof. It is clear that $\mathrm{PT}(S(1, 1, n - 3)) = n - 2$, and $\mathrm{pt}(K_{1,3}) = 2$.

Suppose first that G is a tree such that $\mathrm{PT}(G) = |G| - 2$. Then G is a graph on two parallel paths P_1 and P_2. There is exactly one edge e between the two paths. Observe that e must have an endpoint not in $\{\mathrm{first}(P_i), \mathrm{last}(P_i), i = 1, 2\}$, so without loss of generality $\mathrm{first}(P_1) \neq \mathrm{last}(P_1)$ and neither $\mathrm{first}(P_1)$ nor $\mathrm{last}(P_1)$ is an endpoint of e. If G is a graph with multiple vertices in each of P_1, P_2 (i.e., if $\mathrm{first}(P_2) \neq \mathrm{last}(P_2)$), then no matter which minimum zero forcing set we choose, more than one force will occur at some time. So assume $V(P_2)$ consists of a single vertex w. If the parallel paths were constructed from a minimum zero forcing set B, then $w \in B$, and without loss of generality $B = \{\mathrm{first}(P_1), w\}$. If $N(w) \neq N(\mathrm{first}(P_1))$, then at time one, two vertices would be forced. Thus $N(w) = N(\mathrm{first}(P_1))$ and $G = S(1, 1, n - 3)$.

Now suppose that G is a tree such that $\mathrm{pt}(G) = |G| - 2$. This implies $\mathrm{PT}(G) = |G| - 2$, so $G = S(1, 1, n - 3)$. Since $n - 3 > 1$ implies $\mathrm{pt}(S(1, 1, n - 3)) < n - 2$, $G = S(1, 1, 1) = K_{1,3}$. □

Figure 6: Graphs G and minimum zero forcing sets B such that $\mathrm{pt}(G,B) < |G|-2$ (where light vertices may be absent or repeated and similarly for light edges)

Observation 3.5. *If G is one of the graphs shown in Figure 6, then $\mathrm{pt}(G) < |G|-2$, because the black vertices are a minimum zero forcing set B with $\mathrm{pt}(G,B) < |G|-2$.*

For any graph and vertices x, y, $x \sim y$ denotes that x and y are adjacent, and xy denotes the edge with endpoints x and y.

Definition 3.6. Start with two disjoint paths $P_i, i = 1,2$ (where the edges of $P_i, i = 1,2$ are called *path edges*). The *zigzag path* Q alternates between the two paths P_1 and P_2, starting with P_1. Even though each vertex of Q is a vertex of P_i, for convenience, we also label the vertices of Q by $z_1, z_2, \ldots, z_\ell$ (so $z_{2j} \in V(P_2)$ and $z_{2j+1} \in V(P_1)$). The vertices z_i and edges $z_i z_{i+1}$ of Q are called *zigzag vertices* and *zigzag edges*, and the number of edges of Q is called the *zigzag length*. Let $G_0 = P_1 \cup P_2 \cup Q$. Some (or none or all) of the following additional edges are added to G_0 to obtain G:

$$z_j w \text{ where } 1 < j < \ell, w \in \overline{\mathrm{path}}(z_j), \text{ and } z_{j-1} \prec w \prec z_{j+1}.$$

A graph that can be constructed in this manner with zigzag length at least two is a *zigzag graph*, and the labeling $P_i, i = 1,2$, $Q = (z_1, z_2, z_3, \ldots, z_\ell)$ is called a *zigzag labeling*.

Observe that any zigzag graph is a graph on two parallel paths. Examples of zigzag graphs are shown in Figures 6, 7, and 8.

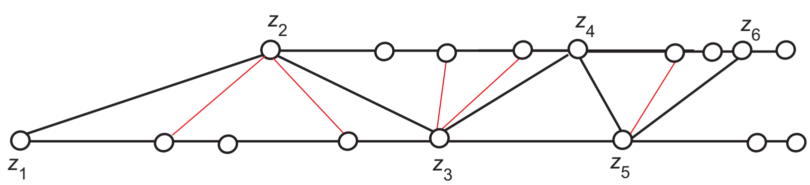

Figure 7: A zigzag graph, with path edges and zigzag edges in black

Theorem 3.7. *Let G be a graph. Then $\mathrm{pt}(G) = |G|-2$ if and only if G is one of the following:*

1. *$P_{n-1} \dot\cup P_1$.*
2. *$K_{1,3}$.*
3. *A zigzag graph with zigzag labeling such that all of the following conditions are satisfied:*
   (a) *G is not isomorphic to one of the graphs shown in Figure 6.*
   (b) *$\deg(\mathrm{first}(P_1)) > 1$ or $\deg(\mathrm{first}(P_2)) > 1$ (both paths cannot begin with vertices having degree one in G).*

(c) $\deg(\text{last}(P_1)) > 1$ *or* $\deg(\text{last}(P_2)) > 1$ *(both paths cannot end with vertices having degree one in* G*).*

(d) $z_2 \neq \text{first}(P_2)$ *or* $\text{first}(P_2) \sim \text{next}(z_1)$

(e) $z_{\ell-1} \neq \text{last}(\text{path}(z_{\ell-1}))$ *or* $\text{last}(\text{path}(z_\ell)) \sim \text{prev}(z_\ell)$

An example of a zigzag graph satisfying conditions (3a) – (3e) is shown in Figure 8.

Proof. Assume $\text{pt}(G) = |G| - 2$. If G is disconnected or a tree, then G is $P_{n-1}\dot{\cup}P_1$ or $K_{1,3}$ by Lemmas 3.3 and 3.4. So assume G is connected and has a cycle.

First we find a zigzag labeling for G. There exists a minimum zero forcing set B of cardinality 2 such that $\text{pt}(G, B) = |G| - 2$, i.e, exactly one force is performed at each time for B. Renumber the vertices of G as follows: vertices $V(G) = \{-1, 0, 1, 2, \ldots, n-2\}$, zero forcing set $B = \{-1, 0\}$ with $0 \to 1$, and vertex t is forced at time t. Then G is a graph on two parallel paths $P_i, i = 1, 2$, where P_1 and P_2 are the two forcing chains (with the path order being the forcing order). Observe that $\deg(0) \leq 2$ and $\deg(-1) \geq 2$, because 0 can immediately force and -1 cannot. If $\deg(-1) = 2$ and $|G| > 3$, then choose P_1 to be $\text{path}(-1)$, and let $z_1 = -1$, $z_2 = \max N(-1) \cap P_2$. Otherwise, choose P_1 to be $\text{path}(0)$ and let $z_1 = \min N(-1)$, $z_2 = -1$. For $j \geq 2$, define $z_{j+1} = \max N(z_j) \cap \overline{\text{path}}(z_j)$ until $N(z_j) \cap \overline{\text{path}}(z_j) = \emptyset$. With this labeling, G is a zigzag graph with zigzag labeling.

Now we show that with this zigzag labeling, G satisfies conditions (3a) – (3e). Since $\text{pt}(G) = |G| - 2$, G is not isomorphic to one of the graphs shown in Figure 6, i.e., condition (3a) is satisfied. Since -1 is the first vertex in one of the paths and $\deg(-1) \geq 2$, condition (3b) is satisfied. The remaining conditions must be satisfied or there is a different zero forcing set of two vertices with lower propagation time: if (3c) fails, use $B = \{\text{last}(P_1), \text{last}(P_2)\}$; if (3d) fails, then $z_2 = \text{first}(P_2)$ and $\text{first}(P_2) \not\sim \text{next}(z_1)$, so use $B = \{\text{first}(P_1), \text{next}(z_1)\}$; if (3e) fails, this is analogous to (3d) failing, use $B = \{\text{last}(\text{path}(z_\ell)), \text{prev}(z_\ell)\}$.

For the converse, $\text{pt}(G) = |G| - 2$ for $G = P_{n-1}\dot{\cup}P_1$ or $G = K_{1,3}$ by Lemmas 3.3 and 3.4. So assume G is a zigzag graph with a zigzag labeling satisfying conditions (3a) – (3e). The sets $B_1 = \{\text{first}(P_1), \text{first}(P_2)\}$ and $B_2 = \{\text{last}(P_1), \text{last}(P_2)\}$ are minimum zero forcing sets of G, and $\text{pt}(G, B_i) = |G| - 2$ for $i = 1, 2$. If $z_2 = \text{first}(P_2)$, so $\text{first}(P_2) \sim \text{next}(z_1)$, then $B_3 = \{\text{first}(P_1), \text{next}(z_1)\}$ is a zero forcing set and $\text{pt}(G, B_3) = |G| - 2$. If $z_{\ell-1} \neq \text{last}(\text{path}(z_{\ell-1}))$, so $\text{last}(\text{path}(z_{\ell-1})) \sim \text{prev}(z_\ell)$, then $B_4 = \{\text{last}(\text{path}(z_\ell)), \text{prev}(z_\ell)\}$ is a zero forcing set and $\text{pt}(G, B_4) = |G| - 2$. If G is not isomorphic to one of the graphs shown in Figure 6, these are the only minimum zero forcing sets. □

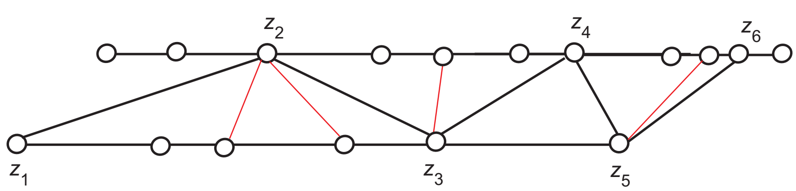

Figure 8: A zigzag graph G having $\text{pt}(G) = |G| - 2$

3.2 Low propagation time

The case of minimum propagation time zero is easy.

Observation 3.8. *For a graph G, the following are equivalent.*

1. $\text{pt}(G) = 0$.

2. $\mathrm{PT}(G) = 0$.

3. $\mathrm{Z}(G) = |G|$.

4. G has no edges.

Next we consider $\mathrm{pt}(G) = 1$. From Observation 1.8 we have the following.

Observation 3.9. *If G is a graph such that $\mathrm{pt}(G) = 1$, then $\mathrm{Z}(G) \geq \frac{|G|}{2}$.*

It is easy to see that the necessary condition in Observation 3.9 is not sufficient.

Example 3.10. Let G be the graph obtained from K_4 by appending a leaf to one vertex. Then $\mathrm{Z}(G) = 3 > |G|/2$ and $\mathrm{pt}(G) = 2$.

Theorem 3.11. *Let $G = (V, E)$ be a graph such that $\mathrm{pt}(G) = 1$. For $v \in V$, $v \in \bigcap_{B \in \mathrm{Eff}(G)} B$ if and only if for every $B \in \mathrm{Eff}(G)$, $|B^{(1)} \cap N(v)| \geq 2$.*

Proof. If for every $B \in \mathrm{Eff}(G)$, $|B^{(1)} \cap N(v)| \geq 2$, then v cannot perform a force in an efficient set of forces, so $v \in \mathrm{Term}(\mathcal{F})$ for every $\mathcal{F} \in \mathcal{F}_{eff}(G)$. Thus $v \in \bigcap_{\mathcal{F} \in \mathcal{F}_{eff}(G)} \mathrm{Term}(\mathcal{F}) = \bigcap_{B \in \mathrm{Eff}(G)} B$.

Now suppose $v \in \bigcap_{B \in \mathrm{Eff}(G)} B$ and let $B \in \mathrm{Eff}(G)$. If v performs a force in an efficient set of forces $\mathcal{F}$ of B, then $v \notin \mathrm{Term}(\mathcal{F}) \in \mathrm{Eff}(G)$, so v cannot perform a force in any such $\mathcal{F}$. Since v cannot perform a force, $|B^{(1)} \cap N(v)| \neq 1$. It is shown in [2] that every vertex of a minimum zero forcing set must have a neighbor not in the zero forcing set, so $|B^{(1)} \cap N(v)| \geq 2$. □

We now consider the case of a graph G that has $\mathrm{pt}(G) = 1$ and $\mathrm{Z}(G) = \frac{1}{2}|G|$. Examples of such graphs include the hypercubes Q_s [1].

Definition 3.12. Suppose $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are graphs of order n and $\mu : V_1 \to V_2$ is a bijection. Define the *matching graph* (H_1, H_2, μ) to be the graph constructed as the disjoint union of H_1, H_2 and the perfect matching between V_1 and V_2 defined by μ.

Matching graphs play a central role in the study of graphs that have propagation time one.

Proposition 3.13. *Let $G = (V, E)$ be a graph. Then any two of the following conditions imply the third.*

1. $|G| = 2\,\mathrm{Z}(G)$.

2. $\mathrm{pt}(G) = 1$.

3. G is a matching graph

Proof. (1) & (2) $\Rightarrow$ (3): Let B be an efficient zero forcing set of G and let $\overline{B} = V \setminus B$. Since $|B| = \frac{1}{2}|G|$ and $\mathrm{pt}(G) = 1$, every element $b \in B$ must perform a force at time one. Thus $|N(b) \cap \overline{B}| = 1$ and there exists a perfect matching between B and $\overline{B}$ defined by $\mu : B \to \overline{B}$ where $\mu(b) \in N(b) \cap \overline{B}$. Then $G = (B, \overline{B}, \mu)$.

For the remaining two parts, assume $G = (H_1, H_2, \mu)$.

(1) & (3) $\Rightarrow$ (2): Since $\mathrm{Z}(G) = n$, H_1 is a minimum zero forcing set and $\mathrm{pt}(G, H_1) = 1$.

(2) & (3) $\Rightarrow$ (1): Since $\mathrm{pt}(G) = 1$, $\mathrm{Z}(G) \geq n$, and $\mathrm{Z}(G) \leq n$ because H_1 is a zero forcing set with $\mathrm{pt}(G, H_1) = 1$. □

We examine conditions that ensure $\mathrm{Z}((H_1, H_2, \mu)) = n$ and thus $\mathrm{pt}((H_1, H_2, \mu)) = 1$. The matching μ affects the zero forcing number and propagation time, as the next two examples show.

Example 3.14. The Cartesian product $C_5\Box P_2$ is (C_5, C_5, ι), where ι is the identity mapping. It is known that $\mathrm{Z}(C_5\Box P_2) = 4$ [1] and thus $\mathrm{pt}(C_5\Box P_2) > 1$.

Example 3.15. The Petersen graph P can be constructed as (C_5, C_5, μ_P) where $\mu_P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix}$. It is known that $\mathrm{Z}(P) = 5$ [1] and thus $\mathrm{pt}(P) = 1$.

Let $c(G)$ denote the number of components of G.

Theorem 3.16. *Let $|H_1| = |H_2| = n$ and let $\mu : H_1 \to H_2$ be a bijection. If $\mathrm{pt}((H_1, H_2, \mu)) = 1$, then $c(H_1) = c(H_2) = c((H_1, H_2, \mu))$.*

Proof. Assume it is not the case that $c(H_1) = c(H_2) = c((H_1, H_2, \mu))$. This implies μ is not the union of perfect matchings between the components of H_1 and the components of H_2. Without loss of generality, there is a component C_1 of H_1 that is not matched within a single component of H_2. Then there exist vertices u and v in C_1 such that $\mu(v) \in C_v$, $\mu(u) \in C_u$, and C_v and C_u are separate components of H_2. We show that there is a zero forcing set of size $n-1$ for (H_1, H_2, μ), and thus $\mathrm{pt}((H_1, H_2, \mu)) > 1$. Let $B_1 = C_1 \backslash (\mu^{-1}(C_v) \bigcup \{u\})$, $B_2 = V_2 \backslash (\mu^{-1}(B_1) \bigcup \mu(u))$, and $B = B_1 \bigcup B_2$. Clearly $|B| = n-1$. Then $x \to \mu^{-1}(x)$ for $x \in C_v$, $w \to v$ for some $w \in C_1$, $y \to \mu(y)$ for $y \in C_1 \backslash \mu^{-1}(C_v)$, and $z \to \mu^{-1}(z)$ for all $\mu^{-1}(z)$ in the remaining components of H_1. Therefore B is a zero forcing set as claimed, so $\mathrm{Z}((H_1, H_2, \mu)) \leq n-1$. Thus $\mathrm{pt}((H_1, H_2, \mu)) > 1$. □

Theorem 3.17. *Let $|H| = n$ and let μ be a bijection of vertices of H and K_n (with μ acting on the vertices of H). Then $\mathrm{pt}((H, K_n, \mu)) = 1$ if and only if H is connected.*

Proof. If H is not connected, then $\mathrm{pt}((H, K_n, \mu)) \neq 1$ by Theorem 3.16. Now assume H is connected and $G = (H, K_n, \mu)$. Let $B \subseteq V(G)$ with $|B| = n-1$. We show B is not a zero forcing set. This implies $\mathrm{Z}(G) = n$ and thus $\mathrm{pt}(G) = 1$. Let $X = V(K_n)$ and $Y = V(H)$. For $x \in X$, x cannot perform a force until at least $n-1$ vertices in X are black. If $|X \cap B| = n-1$ then no force can be performed. So assume $|X \cap B| \leq n-2$. Until at least $n-1$ vertices in X are black, all forces must be performed by vertices in Y. We show that no more than $n-2$ vertices in X can turn black. Perform all forces of the type $y \to y'$ with $y, y' \in Y$. For each such force, $\mu(y)$ must be black already. Thus at most $|X \cap B|$ such forces within Y can be performed. So there are now at most $|Y \cap B| + |X \cap B| = n-1$ black vertices in Y. Note first that if at most $n-2$ vertices of Y are black, then after all possible forces from Y to X are done, no further forces are possible, and at most $n-2$ vertices in X are black. So assume $n-1$ vertices of Y are black. Let $w \in Y$ be white. Since H is connected, there must be a neighbor u of w in Y, and u is black. Since $u \in N(w)$ and w is white, u has not performed a force. If $\mu(u)$ were black, there would be at most $n-2$ black vertices in Y, so $\mu(u)$ is white. After preforming all possible forces from Y to X, at most $n-2$ vertices in X are black because all originally black vertices x have $\mu^{-1}(x)$ black, there are $n-1$ black vertices in Y, and u cannot perform a force at this time (since both w and $\mu(u)$ are white). Thus not more than $n-2$ vertices of X can be forced, and B is not a zero forcing set. □

The Cartesian product of G with P_2 is one way of constructing matching graphs, because $G \,\Box\, P_2 = (G, G, \iota)$. Examples of graphs G having $\mathrm{Z}(G \,\Box\, P_2) = |G|$ include the complete graph K_r and hypercube Q_s [1]. Since $\mathrm{Z}(G \,\Box\, P_2) \leq 2\,\mathrm{Z}(G)$ [1], to have $\mathrm{Z}(G \,\Box\, P_2) = |G|$ it is necessary that $\mathrm{Z}(G) \geq \frac{|G|}{2}$, but that condition is not sufficient.

Example 3.18. Observe that $\mathrm{Z}(K_{1,r}) = r - 1 \geq \frac{1}{2}|K_{1,r}|$ for $r \geq 3$. But $\mathrm{Z}(K_{1,r} \,\Box\, P_2) = r < |K_{1,r}|$, so $\mathrm{pt}(K_{1,r} \,\Box\, P_2) \geq 2$.

The next theorem provides conditions that ensure that iterating the Cartesian product with P_2 gives a graph with propagation time one. Recall that one of the original motivations for defining the zero forcing number was to bound maximum nullity, and the interplay between these two parameters is central to the proof of the next theorem. Let G be a graph. The *set of symmetric matrices described by* G is $\mathcal{S}(G) = \{A \in \mathbb{R}^n : A^T = A \text{ and for } i \neq j, a_{ij} \neq 0 \Leftrightarrow \{i,j\} \in E(G)\}$. The *maximum nullity of* G is $\mathrm{M}(G) = \max\{\operatorname{null} A : A \in \mathcal{S}(G)\}$. It is well known that $\mathrm{M}(G) \leq \mathrm{Z}(G)$ [1]. The next theorem provides conditions that are sufficient to iterate the construction of taking the Cartesian product of a graph and P_2 and obtain minimum propagation time equal to one.

Theorem 3.19. *Suppose G is a graph with $|G| = n$ and there exists a matrix $L \in \mathcal{S}(G)$ such that $L^2 = I_n$. Then*

$$\mathrm{M}(G \,\square\, P_2) = \mathrm{Z}(G \,\square\, P_2) = n \text{ and } \mathrm{pt}(G \,\square\, P_2) = 1.$$

Furthermore, for

$$\hat{L} = \frac{1}{\sqrt{2}} \begin{bmatrix} L & I_n \\ I_n & -L \end{bmatrix}$$

$\hat{L} \in \mathcal{S}(G \,\square\, P_2)$ and $\hat{L}^2 = I_{2n}$.

Proof. Given the $n \times n$ matrix L, define

$$H = \begin{bmatrix} L & I_n \\ I_n & L \end{bmatrix}.$$

Clearly $H, \hat{L} \in \mathcal{S}(G \,\square\, P_2)$ and $\hat{L}^2 = I_{2n}$. Since $\begin{bmatrix} I_n & 0 \\ -L & I_n \end{bmatrix} \begin{bmatrix} L & I_n \\ I_n & L \end{bmatrix} = \begin{bmatrix} L & I_n \\ 0 & 0 \end{bmatrix}$, $\mathrm{null}(H) = n$. Therefore, $\mathrm{M}(G \,\square\, P_2) \geq n$. Then

$$n \leq \mathrm{M}(G \,\square\, P_2) \leq \mathrm{Z}(QG \,\square\, P_2) \leq n$$

so we have equality throughout. Thus the set of vertices in one copy of G is a minimum zero forcing set and each vertex can force independently. Therefore, $\mathrm{pt}(G) = 1$. □

Let $G\,(\square\, P_2)^s$ denote the graph constructed by starting with G and performing the Cartesian product with P_2 s times. For example, the hypercube $Q_s = P_2\,(\square\, P_2)^{s-1}$, and the proof given in [1] that $\mathrm{M}(Q_s) = \mathrm{Z}(Q_s) = 2^{s-1}$ is the same as the proof of Theorem 3.19 using the matrix $L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{S}(P_2)$.

Corollary 3.20. *Suppose G is a graph such that there exists a matrix $L \in \mathcal{S}(G)$ such that $L^2 = I_{|G|}$. Then for $s \geq 1$,*

$$\mathrm{M}(G\,(\square\, P_2)^s) = \mathrm{Z}(G\,(\square\, P_2)^s) = |G|2^{s-1} \text{ and } \mathrm{pt}(G\,(\square\, P_2)^s) = 1.$$

Corollary 3.21. *For $s \geq 1$, $\mathrm{M}(K_n\,(\square\, P_2)^s) = \mathrm{Z}(K_n\,(\square\, P_2)^s) = n2^{s-1}$ and $\mathrm{pt}(K_n\,(\square\, P_2)^s) = 1$.*

Proof. Let $L = \frac{2}{n} J_n - I_n$, where J_n is the $n \times n$ matrix having all entries equal to one. Then $L \in \mathcal{S}(K_n)$ and $L^2 = I_n$. □

We have established a number of constructions that provide matching graphs having propagation time one. For example, $\mathrm{pt}(P_2\,(\square\, P_2)^s) = 1$, $\mathrm{pt}(K_n\,(\square\, P_2)^s) = 1$, and if H is connected, then $\mathrm{pt}((K_n, H, \mu)) = 1$ for every matching μ. But the general question remains open.

Question 3.22. *Characterize matching graphs (H_1, H_2, μ) such that $\mathrm{pt}((H_1, H_2, \mu)) = 1$.*

We can investigate minimum propagation time one by deleting vertices that are in an efficient zero forcing set but do not perform a force in an efficient set of forces. The next result is a consequence of Theorem 2.16 and [1].

Corollary 3.23. *Let G be a graph with $\operatorname{pt}(G) = 1$, B an efficient zero forcing set of G, and $\mathcal{F}$ an efficient set of forces in which v does not perform a force. Then $\operatorname{pt}(G - v) = \operatorname{pt}(G) = 1$.*

Definition 3.24. Let G be a graph with $\operatorname{pt}(G) = 1$, B an efficient zero forcing set of G, $\mathcal{F}$ an efficient set of forces, and S the set of vertices in B that do not perform a force. Define $V' = V \setminus S$, $G' = G[V'] = G - S$, $B' = B \setminus S$, and $\overline{B'} = V' \setminus B$. The graph G' is called a *prime* subgraph of G with associated zero forcing set B'.

Observation 3.25. *Let G be a graph with $\operatorname{pt}(G) = 1$. For the prime subgraph G' and associated zero forcing set B' defined from an efficient zero forcing set B and efficient set of forces $\mathcal{F}$,*

1. $|B|' = |\overline{B'}|$ *and* $|G'| = 2|B'|$.
2. $G' = G(G[B'], G[\overline{B'}], \mu)$ *where* $\mu : B' \to \overline{B'}$ *is defined by* $\mu(b) \in (N(b) \cap \overline{B'})$.
3. B' *and* $\overline{B}'$ *are efficient zero forcing sets of* G'.
4. $\operatorname{pt}(G') = 1$.

It is clear that if $G = (V, E)$ has no isolated vertices, $\operatorname{pt}(G) = 1$, and if $\hat{G}$ is constructed from G by adjoining a new vertex v adjacent to every $u \in V(G)$, then $\operatorname{pt}(\hat{G}) = 1$.

We now return to considering $\bigcap_{B \in \operatorname{Eff}(G)} B$, specifically in the case of propagation time one. We have a corollary of Theorem 3.11.

Corollary 3.26. *Let $G = (V, E)$ be a graph such that $\operatorname{pt}(G) = 1$. If $v \in \displaystyle\bigcap_{B \in \operatorname{Eff}(G)} B$, then $\deg v \geq 4$.*

Proof. Let $v \in \displaystyle\bigcap_{B \in \operatorname{Eff}(G)} B$. Since $\operatorname{pt}(G) = 1$, for any efficient set $\mathcal{F}$ of B, $\operatorname{Term}(B) = B^{(1)}$. By Theorem 3.11, $|B^{(1)} \cap N(v)| \geq 2$, so $|\operatorname{Term}(B) \cap N(v)| \geq 2$. Since $B = \operatorname{Term}(\operatorname{Rev}(\mathcal{F}))$, $|B \cap N(v)| \geq 2$. Thus $\deg v \geq 4$. □

Note that Corollary 3.26 is false without the hypothesis that $\operatorname{pt}(G) = 1$, as the next example shows.

Example 3.27. Let G be the graph in Figure 9. It can be verified that $\operatorname{Z}(G) = 2$ and v is in every efficient zero forcing set.

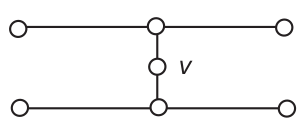

Figure 9: A graph G with $v \in \bigcap_{B \in \operatorname{Eff}(G)} B$ and $\deg v < 4$

Proposition 3.28. *Let $G = (V, E)$ be a graph and $v \in V$. If $\deg v > \operatorname{Z}(G)$ and $\operatorname{pt}(G) = 1$, then $v \in \bigcap_{B \in \operatorname{Eff}(G)} B$.*

Proof. Suppose $v \notin B \in \operatorname{Eff}(G)$, and let $\mathcal{F}$ be an efficient set of forces of B. Then v performs a force in the efficient set $\operatorname{Rev}(\mathcal{F})$ of $\operatorname{Term}(\mathcal{F})$. Since every force is performed at time 1, $\deg v \leq \operatorname{Z}(G)$. □

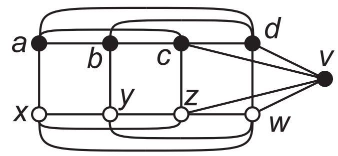

Figure 10: A graph G with $v \in \bigcap_{B \in \mathrm{Eff}(G)} B$ and $\deg v < \mathrm{Z}(G)$

The converse of Proposition 3.28 is false, as the next example demonstrates.

Example 3.29. Let G be the graph in Figure 10. It can be verified that $\mathrm{Z}(G) = 5$. Then $B_1 = \{a, b, c, d, v\}$ and $B_2 = \{x, y, z, w, v\}$ are minimum zero forcing sets and $\mathrm{pt}(G, B_1) = \mathrm{pt}(G, B_2) = 1$, so $\mathrm{pt}(G) = 1$. Let B be a zero forcing set of G not containing vertex v. In order to have $\mathrm{pt}(G, B) = 1$, some neighbor of v must be able to force v immediately. Without loss of generality, this neighbor is d. Then $a, b, c, d, w \in B$. The set $\{a, b, c, d, w\}$ is a minimum zero forcing set but has propagation time 2, because no vertex can force z immediately. Thus $\bigcap_{B \in \mathrm{Eff}(G)} B = \{v\}$, and observe that $\deg v = 4 < 5 = \mathrm{Z}(G)$.

4 Relationship of propagation time and diameter

In general, the diameter and the propagation time of a graph are not comparable. Let G be the dart (shown in Figure 2). Then $\mathrm{diam}(G) = 2 < \mathrm{pt}(G) = 3$. On the other hand, $\mathrm{diam}(C_4) = 2 > 1 = \mathrm{pt}(C_4)$.

Although it is not possible to a obtain a relationship between diameter and propagation time in an arbitrary graph, diameter serves as an upper bound for propagation time in the family of trees. To demonstrate this, we need some definitions. The *walk* $v_1 v_2 \dots v_p$ in G is the subgraph with vertex set $\{v_1, v_2, \dots, v_p\}$ and edge set $\{v_1v_2, v_2v_3, \dots, v_{p-1}v_p\}$ (vertices and/or edges may be repeated in these lists but are not repeated in the vertex and edge sets). A *trail* is a walk with no repeated edges (vertices may be repeated). A *path* is a trail with no repeated vertices. The *length* of a trail P, denoted by $\mathrm{len}(P)$, is the number of edges in P. We show in Lemma 4.1 below that for any graph G and minimum zero forcing set B, there is a trail of length at least $\mathrm{pt}(G, B)$. A trail produced by the method in the proof is illustrated in the Example 4.2 below.

Lemma 4.1. *Let G be a graph and let B be a minimum zero forcing set of G. Then there exists a trail P such that* $\mathrm{pt}(G, B) \le \mathrm{len}(P)$.

Proof. Observe that if $u, v \in V(G)$ such that u forces v at time $t > 1$, then u cannot force v at time $t - 1$. Thus either u was forced at time $t - 1$ or some neighbor of u was forced at time $t - 1$. So there is a path wuv, where w forces u at time $t - 1$, or a path $wxuv$, where w forces x at time $t - 1$ and x is a neighbor of u.

We construct a trail $v_{-p}v_{-p+1}v_{-p+2} \dots v_0$, such that for each time t, $1 \le t \le \mathrm{pt}(G, B)$, there exists an i_t, $-p \le i_t \le -1$, such that v_{i_t} forces v_{i_t+1} at time t. Begin with $t = \mathrm{pt}(G, B)$ and work backwards to $t = 1$ to construct the trail, using negative numbering. To start, there is some vertex v_0 that is forced by a vertex v_{-1} at time $t = \mathrm{pt}(G, B)$; the trail is now $v_{-1}v_0$. Assume the trail $v_{-j} \dots v_0$ has been constructed so that for each time $t = \ell, \dots, \mathrm{pt}(G, B)$, there exists an i_t, such that v_{i_t} forces v_{i_t+1} at time t. If $\ell > 1$, then $v_{-j} \to v_{-j+1}$ at $t = \ell$. Thus either $v_{-j-1}v_{-j}v_{-j+1}$ or $v_{-j-2}v_{-j-1}v_{-j}v_{-j+1}$ is a path in G, and we can extend our trail to $v_{-j-1}v_{-j} \dots v_0$ or $v_{-j-2}v_{-j-1}v_{-j} \dots v_0$. It should be obvious that no forcing edge will appear in this walk multiple times (by our construction). If uv is not a forcing edge, then it can only appear in our walk if u' forced u and v forced v'. Let $u'uvv'$ be the first occurrence of uv in our walk. If uv were to occur again, either u or v would need to be forced at this later time, but this cannot happen because u and v are both black at this point. $\square$

Example 4.2. Let G be the graph in Figure 11. As shown by the numbering in the figure, $\mathrm{pt}(G) = 4$, but G does not contain a path of length 4. The trail produced by the method of proof used in Lemma 4.1 is $abcdecf$ and has length 6.

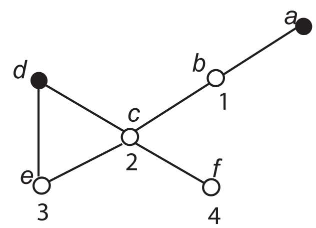

Figure 11: A graph G that does not have a path of length $\mathrm{pt}(G)$

Theorem 4.3. *Let T be a tree and B be a minimum zero forcing set of T. Then $\mathrm{PT}(T, B) \leq \mathrm{diam}(T)$. Hence, $\mathrm{pt}(T) \leq \mathrm{PT}(T) \leq \mathrm{diam}(T)$.*

Proof. Choose B to be a minimum zero forcing set such that $\mathrm{pt}(T, B) = \mathrm{PT}(T)$. By Lemma 4.1, there exists a trail in T of length at least $\mathrm{pt}(T, B)$. Since between any two vertices in a tree there is a unique path, any trail is a path and the diameter of T must be the length of the longest path in T. Therefore,

$$\mathrm{pt}(T) \leq \mathrm{PT}(T) = \mathrm{pt}(T, B) \leq \mathrm{diam}(T). \qquad \square$$

The diameter of a graph G can get arbitrarily larger than its minimum propagation time. The next example exhibits this result, but first we need an observation.

Observation 4.4. *Let G be a graph having exactly ℓ leaves. Since at most two leaves can be on a forcing chain, $\mathrm{Z}(G) \geq \lceil \frac{\ell}{2} \rceil$.*

Example 4.5. To construct a k-comb, we append a leaf to each vertex of a path on k vertices, as shown in Figure 12. Let G denote a k-comb where $k \equiv 0 \mod 4$. It is clear that $\mathrm{diam}(G) = k + 1$. If we number the leaves in path order starting with one, then the set B consisting of every leaf whose number is congruent to 2 or 3 $\mod 4$ (shown in black in the Figure 12) is a zero forcing set, and $|B| = \frac{k}{2}$. Since $\mathrm{Z}(G) \geq \frac{k}{2}$, B is a minimum zero forcing set. Then $\mathrm{pt}(G) \leq \mathrm{pt}(G, B) = 3$. Since $|G| = 2k$, $\mathrm{Z}(G) = \frac{k}{2}$, and $\mathrm{pt}(G) \geq \frac{|G| - \mathrm{Z}(G)}{\mathrm{Z}(G)}$, $\mathrm{pt}(G) \geq 3$. Therefore, $\mathrm{pt}(G) = 3$. Thus the $\mathrm{diam}(G) = k + 1$ is arbitrarily larger than $\mathrm{pt}(G) = 3$.

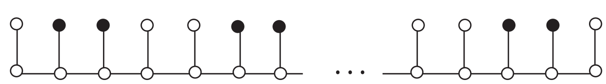

Figure 12: A comb

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