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## Graphs of arbitrary excessive class

G. Mazzuoccolo<sup>a,\*</sup>, M. Young<sup>b,1</sup><sup>a</sup> Dipartimento di Scienze e Metodi dell'Ingegneria, Università di Modena e Reggio Emilia, via Amendola 2, 42100 Reggio Emilia, Italy<sup>b</sup> Department of Mathematics, Iowa State University, Ames, IA 50010, USA

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## ABSTRACT

We show that there exists a family of  $r$ -regular graphs of arbitrarily large excessive index for each integer  $r$  greater than 3. Furthermore, we answer a question in Bonisoli and Cariolaro (2007) [1] showing that all the positive integers can be attained as excessive classes of regular graphs.

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## 1. Introduction

All graphs considered here are simple, finite and undirected (unless otherwise noted). We denote by  $V(G)$  and  $E(G)$  the vertex-set and the edge-set of a graph  $G$ , respectively. A 1-factor of a graph is a collection of independent edges, which together are incident on all the vertices of the graph. Following [1], we use the term *excessive factorization* to denote a minimum cover of the edge-set of a graph by a set of (not necessarily distinct) 1-factors and we shall denote by  $\chi'_e(G)$  the cardinality of such a set, if it does exist. We remark that  $\chi'_e(G)$  is called the excessive index of  $G$  in [1], whereas the same parameter is called the perfect matching index in [2]. Let  $G$  be an  $r$ -regular graph. The excessive class of  $G$  is defined as

$$\text{exc}(G) = \chi'_e(G) - r.$$

A regular graph of excessive class equal to  $n$  for each even integer  $n$  is shown in the proof of Theorem 4.1 in [1]. In the same paper Bonisoli and Cariolaro ask for a parallel construction showing that all the odd integers can also be attained as excessive classes of regular graphs. In this note we exhibit such a construction. In particular, we are able to construct (see Theorem 1) graphs with arbitrary excessive class, so covering in a unique construction both even and odd cases. Nevertheless, we have to remark that the construction given in [1] for the even case results to be a bit “cheaper” than ours in terms of the order of the graph.

The well-known conjecture of Berge and Fulkerson [3] can be stated as follows (see [4]).

**Conjecture 1** (Berge–Fulkerson). *Let  $G$  be a bridgeless cubic graph. Then  $\chi'_e(G) \leq 5$ .*

The following weakening (also suggested by Berge) is still open.

**Conjecture 2.** *There exists a fixed integer  $k$  such that  $\chi'_e(G) \leq k$  for every bridgeless cubic graph.*

We prove that for  $r$ -regular graphs with  $r > 3$  the analogous of the second conjecture is false as well.

\* Corresponding author.

E-mail address: [mazzuoccolo@unimore.it](mailto:mazzuoccolo@unimore.it) (G. Mazzuoccolo).

<sup>1</sup> Worked completed as a Mellon Fellow at Smith College.

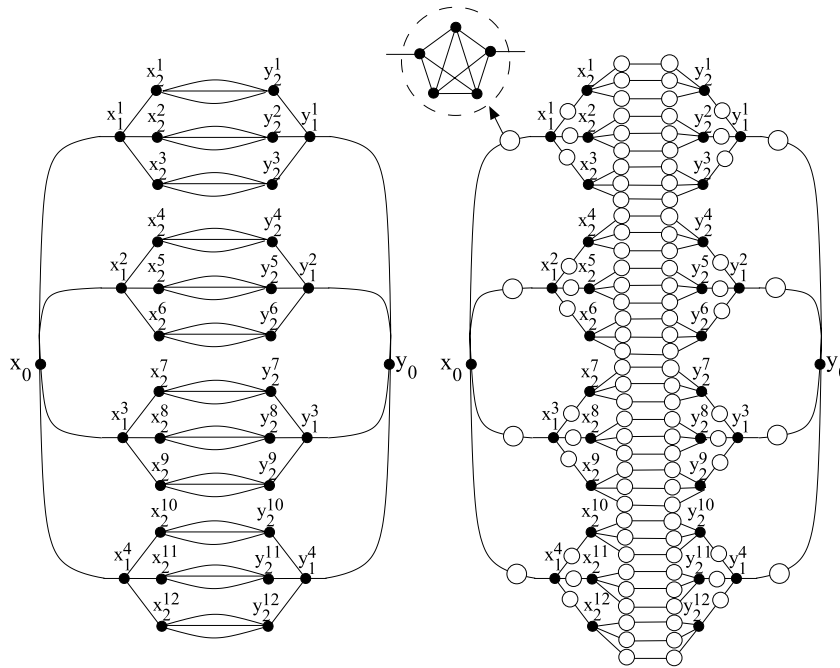


Fig. 1. The multigraph  $G_4^*(2)$  and the graph  $G_4(2)$ .

## 2. $r$ -regular graphs of arbitrary large excessive class

In this section we construct a family of  $r$ -regular graphs with excessive index arbitrarily large. We treat the cases of even and odd  $r$  separately. If  $E$  is a set and  $k$  a positive integer, we shall denote by  $kE$  the multiset in which each element of  $E$  has multiplicity  $k$ .

### 2.1. $r$ even

For each  $r$  even and  $k \geq 1$ , define the multigraph  $G_r^*(k)$  as follows:

$$V(G_r^*(k)) = X \cup Y$$

where

$$X = \{x_0\} \cup \{x_i^j, : i = 1, \dots, k, j = 1, \dots, r(r-1)^{i-1}\},$$

$$Y = \{y_0\} \cup \{y_i^j, : i = 1, \dots, k, j = 1, \dots, r(r-1)^{i-1}\}.$$

$$E(G_r^*(k)) = \begin{cases} E_0 \cup (r-1)E_1 & \text{if } k = 1 \\ E_0 \cup E_2 \cup (r-1)E_1 & \text{if } k > 1 \end{cases}$$

where

$$E_0 = \{[x_0, x_1^h], [y_0, y_1^h] : h = 1, \dots, r\},$$

$$E_1 = \{[x_k^j, y_k^j] : j = 1, \dots, r(r-1)^{k-1}\},$$

$$E_2 = \{[x_i^j, x_{i+1}^h], [y_i^j, y_{i+1}^h], i = 1, \dots, k-1, j = 1, \dots, r(r-1)^{i-1}, h = (r-1)(j-1) + 1, \dots, (r-1)j\}.$$

By  $G_r^*(k)$  we construct the graph  $G_r(k)$  gluing a copy of the complete graph  $K_{r+1}$  minus an edge on each edge of  $G_r^*(k)$  with two vertices in  $X$  or two vertices in  $Y$ , and two copies on each edge with a vertex in  $X$  and a vertex in  $Y$  (see Fig. 1). In what follows we will denote by  $E(X, Y)$  the set of edges of  $G_r(k)$  having both vertices in distinct copies of  $K_{r+1}$  minus an edge.

It is straightforward that  $G_r(k)$  is a simple  $r$ -regular graph. Furthermore, the following proposition holds:

**Proposition 1.**  $\chi'_e(G_r(k)) \geq r(r-1)^{k+1}$ .

**Proof.** Each copy of the complete graph  $K_{r+1}$  minus an edge is connected by two edges to the rest of the graph. Since  $r+1$  is odd, exactly one of those two edges lies in each 1-factor of  $G_r(k)$ . Having that in mind it is easy to verify that each 1-factor covers all the edges in  $E(X, Y)$  but one. Let  $\mathcal{F}$  be a 1-factor cover of  $G_r(k)$  and denote by  $p$  its cardinality. The cardinality of

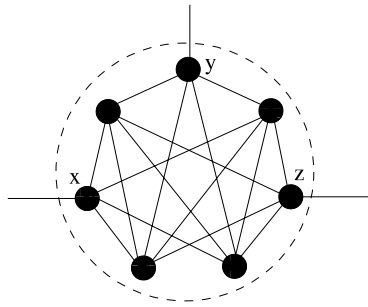


Fig. 2. A-gadget for  $r = 5$ .

$E(X, Y)$  is  $r(r - 1)^k$  then there is at least an edge  $e$  in  $E(X, Y)$  belonging to  $\left\lceil \frac{r(r-1)^k - 1}{r(r-1)^k} p \right\rceil$  1-factors of  $\mathcal{F}$ . Since there are at least  $r - 1$  1-factors of  $\mathcal{F}$  that do not contain  $e$ , we have:

$$\left\lceil \frac{r(r - 1)^k - 1}{r(r - 1)^k} p \right\rceil + r - 1 \leq p$$

that is  $p \geq r(r - 1)^{k+1}$ .  $\square$

Proposition 1 is sufficient to confirm that there exist  $r$ -regular graphs of excessive index arbitrary large for any fixed even  $r > 2$ . For the sake of completeness we establish the exact value of  $\chi'_e(G_r(k))$ .

Let  $G$  be a  $(2, r)$ -graph, that is a graph in which each vertex has degree 2 or  $r$ . If  $x$  is a vertex of  $G$  of degree 2, let  $x_1$  and  $x_2$  be the two neighbours of  $x$  in  $G$ , we shall denote by  $G_x$  the graph  $(G \setminus \{x\}) \cup (K_{r+1} \setminus \{[a, b]\}) \cup \{[x_1, a], [x_2, b]\}$ , where  $a, b$  are two vertices of  $K_{r+1}$ .

**Lemma 1.** *Let  $G$  be a  $(2, r)$ -graph admitting a 1-factor cover  $\mathcal{F}$  of order  $k$ . Suppose there exists a vertex  $x$  of degree 2 such that both  $[x, x_1]$  and  $[x, x_2]$  belong to at least  $r - 1$  1-factors of  $\mathcal{F}$ . Then  $G_x$  admits a 1-factor cover of order  $k$ .*

**Proof.** We show that we can extend each 1-factor of  $G$  in  $\mathcal{F}$  to a 1-factor of  $G_x$ , covering each edge in  $K_{r+1} \setminus \{[a, b]\}$ . Let  $F_1, \dots, F_{r-1}$  be  $r - 1$  of the 1-factors of  $\mathcal{F}$  in which is contained the edge  $[x, x_1]$ . Let  $K_r$  be the complete subgraph of  $G_x$  obtained by  $K_{r+1} \setminus \{[a, b]\}$  removing the vertex  $a$ . The subgraph  $K_r$  has  $r - 1$  disjoint 1-factors, then we can complete each 1-factor  $F_i$  of  $G$  to a 1-factor of  $G_x$  adding the edges in a 1-factor of  $K_r$ . Repeat the same argument on  $r - 1$  of the 1-factors of  $\mathcal{F}$  which contain the edge  $[x, x_2]$ . All the other 1-factors in  $\mathcal{F}$  can be arbitrarily completed to a 1-factor of  $G_x$ . Then we have constructed a 1-factor cover of  $G_x$  with  $k$  1-factors.  $\square$

**Corollary 1.**  $\chi'_e(G_r(k)) = r(r - 1)^{k+1}$ .

**Proof.** Due to Proposition 1 it is sufficient to furnish a 1-factor cover of  $G_r(k)$  with  $r(r - 1)^{k+1}$  factors. Consider the  $(2, r)$ -graph  $H_r(k)$  obtained by  $G_r(k)$  shrinking each copy of  $K_{r+1}$  minus an edge to a vertex. It is an easy check that the graph  $H_r(k)$  has exactly  $r(r - 1)^k$  1-factors. Consider the 1-factor cover  $\mathcal{F}$  of  $H_r(k)$  in which we consider the set of all 1-factors of  $H_r(k)$  repeated  $r - 1$  times. The 1-factor cover  $\mathcal{F}$  has cardinality  $r(r - 1)^{k+1}$  and satisfies the hypothesis of Lemma 1 in each vertex of degree 2 of  $H_r(k)$ . By repeating the application of Lemma 1 on each vertex of  $H$  we obtain the assertion.  $\square$

2.2.  $r > 3$  odd

The basic idea of the odd case is similar to the previous one. We only need some more technicalities.

First we describe two gadgets that will be useful to construct our examples. The gadget of type A is a complete graph  $K_{r+2}$  from which we have removed a triangle with vertices  $x, y, z$  and a 1-factor of the complete graph induced by the remaining vertices. Finally, we add three semiedges with end vertices  $x, y,$  and  $z$  (see Fig. 2).

The gadget of type B has a vertex of degree  $r$  (the black one in Fig. 3) and  $\frac{r-1}{2}$  triples of A-gadgets linked as in the figure. We will picture a B-gadget with a rectangle with a semiedge exiting from a side and  $\frac{r-1}{2}$  semiedges on the opposite side.

Now we are able to describe the graph  $G_r(k)$  for any  $k \geq 1$  and odd  $r$  greater than 3. We define the multigraph  $G_r^*(k)$  as follows:

$$V(G_r^*(k)) = X \cup Y$$

where

$$X = \{x_0\} \cup \left\{ x_i^j, : i = 1, \dots, k, j = 1, \dots, r \left( \frac{r-1}{2} \right)^{i-1} \right\},$$

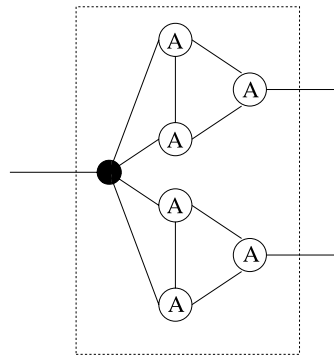


Fig. 3. B-gadget for  $r = 5$ .

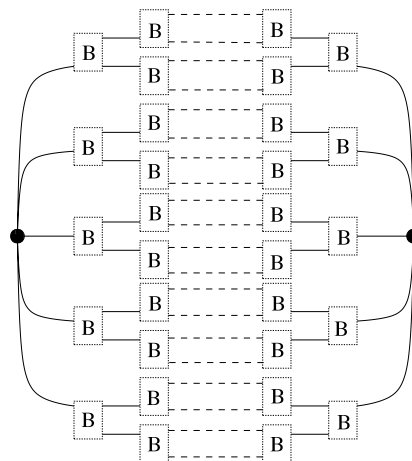


Fig. 4. The graph  $G_5(2)$ .

$$Y = \{y_0\} \cup \left\{ y_i^j : i = 1, \dots, k, j = 1, \dots, r \binom{r-1}{2}^{i-1} \right\}.$$

$$E(G_r^*(k)) = \begin{cases} E_0 \cup \frac{r-1}{2} E_1 & \text{if } k = 1 \\ E_0 \cup E_2 \cup \frac{r-1}{2} E_1 & \text{if } k > 1 \end{cases}$$

where

$$E_0 = \{[x_0^1, x_1^h], [y_0^1, y_1^h] : h = 1, \dots, r\}$$

$$E_1 = \left\{ [x_k^j, y_k^j] : j = 1, \dots, r \binom{r-1}{2}^{k-1} \right\}$$

$$E_2 = \left\{ [x_i^j, x_{i+1}^h], [y_i^j, y_{i+1}^h], i = 1, \dots, k-1, j = 1, \dots, r \binom{r-1}{2}^{i-1}, h = \frac{r-1}{2}(j-1) + 1, \dots, \frac{r-1}{2}j \right\}.$$

Let  $G_r(k)$  be the  $r$ -regular graph obtained by  $G_r^*(k)$  replacing each vertex  $x_i^j$  and  $y_i^j$ ,  $i > 0$ , with a B-gadget as shown in Fig. 4.

**Proposition 2.**  $\chi'_e(G_r(k)) \geq r \binom{r-1}{2}^k$ .

**Proof.** Each copy of the A-gadget is connected by three edges to the rest of the graph, since each A-gadget has an odd number of vertices then exactly one or three of these edges are contained in each 1-factor of  $G_r(k)$ . Now consider a B-gadget. It is connected to the rest of the graph by a set of  $\frac{r+1}{2}$  edges, we will call  $E_B$ . Exactly one of them has the black vertex of Fig. 3 as an end vertex. It is an easy check that if such edge belongs to a 1-factor  $F$  of  $G_r(k)$  then all the other edges of  $E_B$  belong to  $F$ , whereas if it does not belong to  $F$  then exactly one of the others does not belong to  $F$ .

Let  $E$  be the set of  $r \binom{r-1}{2}^k$  edges that have vertices both in distinct B-gadgets and in distinct A-gadgets (in Fig. 4 the set  $E$  is dashed). By the same analysis of B-gadgets, any 1-factor  $F$  of  $G_r(k)$  must contain exactly  $r \binom{r-1}{2}^k - (r-1)$  edges of  $E$ .

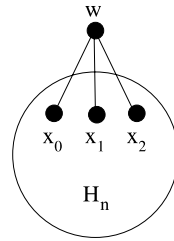


Fig. 5. The graph  $H_n^*$ .

Denoting by  $p$  the cardinality of a 1-factor cover  $\mathcal{F}$  of  $G_r(k)$ , then there is at least an edge  $e$  in  $E$  belonging to  $\lceil \frac{r(\frac{r-1}{2})^k - (r-1)}{r(\frac{r-1}{2})^k} p \rceil$  1-factors of  $\mathcal{F}$ .

Since there are at least  $r - 1$  1-factors of  $\mathcal{F}$  that do not contain  $e$ , we have:

$$\left\lceil \frac{r \left(\frac{r-1}{2}\right)^k - (r-1)}{r \left(\frac{r-1}{2}\right)^k} p \right\rceil + r - 1 \leq p,$$

which implies  $r \left(\frac{r-1}{2}\right)^k \geq p$ .  $\square$

As in the even case, it is possible to establish that the lower bound in Proposition 2 is the best possible. We prefer to omit the proof since it is only a technical variation of the proof of Corollary 1.

**Corollary 2.**  $\chi'_e(G_r(k)) = r \left(\frac{r-1}{2}\right)^{k+1}$ .

We remark that  $\text{exc}(G_r(k))$  is an even integer when  $r$  is even or  $r \equiv 3 \pmod{4}$ , whereas it is odd for  $r \equiv 1 \pmod{4}$ . In the latter case  $\text{exc}(G_r(k)) = r \left[\left(\frac{r-1}{2}\right)^{k+1} - 1\right]$  and then the excessive class of the graphs  $G_r(k)$  does not assume all odd integers values. That leaves open the question posed in [1] about the existence of a graph  $G_n$  with excessive class  $2n + 1$  for each positive integer  $n$ . In the next section we show that all positive integers can be attained as excessive class of regular graphs.

### 3. Graphs of arbitrary excessive class

In what follows we shall denote by  $K_X$  the complete graph on a set  $X$  of vertices and by  $K_{X,Y}$  the bipartite complete graphs with bipartition sets  $X$  and  $Y$ . Let  $n$  be a positive integer,  $n > 2$ , and let  $H_n$  be the graph defined as follows:

$$V(H_n) = \{x_0, x_1, x_2\} \cup U \cup V$$

where  $U = \{u_i : i = 0, \dots, n - 1\}$  and  $V = \{v_i : i = 0, \dots, n - 1\}$ .

$$E(H_n) = E(K_{V(H_n)}) \setminus \{[x_0, x_1], [x_1, x_2], [x_2, x_0]\} \cup \{[u_i, v_i] : i = 0, \dots, n - 1\}.$$

Let  $H_n^*$  be the graph having  $V(H_n^*) = V(H_n) \cup \{w\}$  and  $E(H_n^*) = E(H_n) \cup \{[w, x_i] : i = 1, 2, 3\}$  (see Fig. 5).

Before proving Lemma 2, we need to recall that a near 1-factorization of a finite graph is a partition of the edge-set into near 1-factors, that is into sets of independent edges which cover all vertices but one exactly once. Furthermore, a 1-factor  $F$  will be said orthogonal to a 1-factorization  $\mathcal{F}$  if no pair of edges of  $F$  belongs to the same 1-factor of  $\mathcal{F}$ .

**Lemma 2.**

$$\chi'_e(H_n^*) = 3n.$$

**Proof.** We first prove the inequality  $\chi'_e(H_n^*) \geq 3n$ . Since exactly one of the three edges  $[w, x_0]$ ,  $[w, x_1]$  and  $[w, x_2]$  belongs to each 1-factor of  $H_n^*$ , we may assume that  $[w, x_0]$  appears in at least  $\lceil \frac{\chi'_e(H_n^*)}{3} \rceil$  1-factors. The vertex  $x_0$  has degree  $2n + 1$ , then at least  $2n$  1-factors does not contain the edge  $[w, x_0]$ , one for each edge incident  $x_0$  other than  $[w, x_0]$ . This proves that

$$\chi'_e(H_n^*) \geq \left\lceil \frac{\chi'_e(H_n^*)}{3} \right\rceil + 2n$$

which gives

$$\chi'_e(H_n^*) \geq 3n.$$

Now we exhibit a 1-factor cover of  $H_n^*$  of size  $3n$  to prove that  $\chi'_e(H_n^*) \leq 3n$ . Let us consider separately the cases of odd and even  $n$ . Let  $n$  be an odd integer.

Let  $\mathcal{F}_U(\mathcal{F}_V)$  be a near 1-factorization of  $K_U(K_V)$ , and we denote by  $F'_i(F''_i)$  the near 1-factor of  $\mathcal{F}_U(\mathcal{F}_V)$  leaving the vertex  $u_i(v_i)$  uncovered. Furthermore, let  $\mathcal{F}_{U,V}$  be a 1-factorization of the bipartite complete graph  $K_{U,V}$  such that the 1-factor  $\{[u_i, v_i] : i = 0, \dots, n - 1\}$  is orthogonal to  $\mathcal{F}_{U,V}$ . We denote by  $F_i$  the unique 1-factor of  $\mathcal{F}_{U,V}$  containing the edge  $[u_i, v_i]$ , note that this definition is well-posed by the existence of the orthogonal 1-factor  $\{[u_i, v_i] : i = 0, \dots, n - 1\}$ .

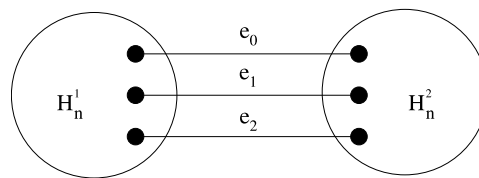


Fig. 6. The graph  $G_n$ .

Consider the following 1-factors of  $H_n^*$ :

- $A_i = \{[w, x_0], [x_1, u_i], [x_2, v_i]\} \cup \{F_i \setminus \{[u_i, v_i]\}\}$ ,  $i = 0, \dots, n - 1$ .
- $B_i = \{[w, x_1], [x_2, u_i], [x_0, v_i]\} \cup F'_i \cup F''_i$ ,  $i = 0, \dots, n - 1$ .
- $C_i = \{[w, x_2], [x_0, u_i], [x_1, v_i]\} \cup F'_i \cup F''_i$ ,  $i = 0, \dots, n - 1$ .

The set  $\cup_{i=0}^{n-1} (A_i \cup B_i \cup C_i)$  is a 1-factor cover of  $H_n^*$  of size  $3n$ .

Let  $n$  be an even integer. Let  $\mathcal{F}_U(\mathcal{F}_V)$  be a 1-factorization of  $K_U(K_V)$ , and we denote by  $F'_i(F''_i)$ ,  $i = 0, \dots, n - 2$ , a 1-factor of  $\mathcal{F}_U(\mathcal{F}_V)$ . As before, let  $\mathcal{F}_{U,V}$  be a 1-factorization of the bipartite complete graph  $K_{U,V}$  such that the 1-factor  $\{[u_i, v_i] : i = 0, \dots, n - 1\}$  is orthogonal to  $\mathcal{F}_{U,V}$ . We also maintain the same notation for a 1-factor  $F_i$  of  $\mathcal{F}_{U,V}$ .

Consider the following 1-factors of  $H_n^*$ :

- $A_i = \{[w, x_0], [x_1, u_i], [x_2, v_i]\} \cup \{F_i \setminus \{[u_i, v_i]\}\}$ ,  $i = 0, \dots, n - 1$ .
- $B_i = \{[w, x_1], [x_2, u_i], [x_0, u_{i+1}]\} \cup F''_i \cup \{\text{An arbitrary 1-factor of } K_{U \setminus \{u_i, u_{i+1}\}}\}$ ,  $i = 0, \dots, n - 2$ .
- $C_i = \{[w, x_2], [x_0, v_i], [x_1, v_{i+1}]\} \cup F'_i \cup \{\text{An arbitrary 1-factor of } K_{V \setminus \{v_i, v_{i+1}\}}\}$ ,  $i = 0, \dots, n - 2$ .
- $D = \{[w, x_1], [x_2, u_{n-1}], [x_0, u_0]\} \cup \{\text{An arbitrary 1-factor of } K_V\} \cup \{\text{An arbitrary 1-factor of } K_{U \setminus \{u_0, u_{n-1}\}}\}$ .
- $E = \{[w, x_2], [x_0, v_{n-1}], [x_1, v_0]\} \cup \{\text{An arbitrary 1-factor of } K_U\} \cup \{\text{An arbitrary 1-factor of } K_{V \setminus \{v_0, v_{n-1}\}}\}$ .

The set  $\cup_{i=0}^{n-1} (A_i \cup B_i \cup C_i) \cup D \cup E$  is a 1-factor cover of  $H_n^*$  of size  $3n$ . This completes the proof of our assertion.  $\square$

Let  $G_n$  be the  $(2n + 1)$ -regular graph obtained by gluing together two copies of  $H_n$  as in Fig. 6.

**Theorem 1.** For each integer  $n > 2$ ,  $exc(G_n) = n - 1$ .

**Proof.** Since  $|V(H_n)|$  is odd then at least one of the edges  $e_0, e_1, e_2$  belong to each 1-factor of  $G_n$ . Repeating the same argument used in Lemma 2 for the graph  $H_n^*$  we obtained  $\chi'_e(G_n) \geq 3n$ . Furthermore, consider the near 1-factors of  $H_n^j$  obtained by the 1-factors of  $H_n^*$  (as given in Lemma 2) removing the edges  $[w, x_i]$ . Let  $G$  be the union of two near 1-factors, one for each subgraph  $H_n^j$ , and such that both of them have the vertex  $x_i$  uncovered. Adding to  $G$  the edge  $e_i$  yields to a 1-factor of  $G_n$ . The union of all the  $3n$  1-factors of  $G_n$  obtained in this way produces a 1-factor cover of  $G_n$ . This proves  $\chi'_e(G_n) \leq 3n$ . Since  $G_n$  is a regular graph of degree  $2n + 1$ , we obtain  $exc(G_n) = 3n - (2n + 1) = n - 1$ .  $\square$

#### 4. Final remarks

If  $X \subset V(G)$ , let  $\partial(X)$  be the set of edges incident with exactly one vertex of  $X$ . An  $r$ -graph is defined as an  $r$  regular graph such that  $|\partial(X)| \geq r$ , for every nontrivial set  $X \subset V(G)$  of odd size. In [5], Seymour defined  $r$ -graph and stated the Berge–Fulkerson in a general form.

**Conjecture 3 (Generalized Berge–Fulkerson).** Let  $G$  be an  $r$ -graph. Then there exist  $2r$  1-factors of  $G$  with the property that every edge of  $G$  is contained in exactly two of the 1-factors.

This conjecture would imply that for any  $r$ -graph  $G$ ,  $\chi'_e(G) \leq 2r - 1$ . The class of graphs constructed in Section 2 have arbitrarily large excessive index; however, we would like to remark that these graphs are not  $r$ -graphs, thus do not contradict the generalized Berge–Fulkerson conjecture.

Furthermore, in the case,  $r = 4$ , removing the edges of any 1-factor of  $G_4(k)$  yields a 1-connected cubic graph, so there exists an edge not belonging to any 1-factor of the graph. If this were not the case,  $G_4(k)$  minus a 1-factor would be a counterexample to the Berge–Fulkerson conjecture.

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