1-factor covers of regular graphs

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Abstract

We consider minimal 1-factor covers of regular multigraphs, focusing on those that are 1-factorizations. In particular, we classify cubic graphs such that every minimal 1-factor cover is also a 1-factorization, and also classify simple regular bipartite graphs with this property. For $r > 3$, we show that there are finitely many simple r-regular graphs such that every minimal 1-factor cover is also a 1-factorization.

1 Introduction

A *perfect matching*, or *1-factor*, of a graph G is a 1-regular spanning subgraph of G. Perfect matchings of graphs have been studied widely and deeply (witness [5]). Let $\mathcal{F}$ be a set of 1-factors of graph G. We say $\mathcal{F}$ *covers* $e \in E(G)$ if there exists a 1-factor $F \in \mathcal{F}$ such that $e \in F$. We say $\mathcal{F}$ is a *1-factor cover* of G if $\mathcal{F}$ covers every edge in G.

A *1-factorization* of G is a 1-factor cover $\mathcal{F}$ in which each edge of G is contained in exactly one 1-factor of $\mathcal{F}$. The generalized Berge-Fulkerson conjecture [6] considers 1-factor covers in which each edge of a graph G is contained in exactly two 1-factors.

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We will mainly be concerned with minimal 1-factor covers. A 1-factor cover $\mathcal{F}$ is *minimal* if no proper subset of $\mathcal{F}$ is also a 1-factor cover.

An *excessive factorization* of G is a minimal 1-factor cover of G with minimum size. Excessive factorizations were introduced in [2]. In order for a graph G to have an excessive factorization (or 1-factor cover), every edge of G must be contained in some 1-factor of G. A graph G is *1-extendable* (or *matching covered*) if every edge of G is contained in a 1-factor of G. A 1-factor of G is *extendable* if it is contained in an excessive factorization of G. Define $EF(G)$ as the set of extendable 1-factors of G and for each $e \in E(G)$ define $EF_G(e)$ as the set of extendable 1-factors of G containing e.

Let G be a 1-extendable graph. The *excessive index* of G, denoted $\chi'_e(G)$, is the size of an excessive factorization. If G is not 1-extendable, e.g. if G is an odd cycle, $\chi'_e(G) = \infty$.

Let G be an r-regular graph of even order. The *excessive class* of G is defined by $exc(G) = \chi'_e(G) - r$. It should be noted that this definition extends to nonregular graphs by replacing r with the largest degree in G [1]. However, throughout this paper we will only be concerned with the excessive classes of regular graphs.

Let $\mathcal{F}_{max}$ be a minimal 1-factor cover of G of maximum size. Then we define $exc_{max}(G) = |\mathcal{F}_{max}| - r$.

We are interested in graphs G for which every minimal 1-factor cover is a 1-factorization. In order for G to have a 1-factorization, G must be r-regular and r-edge colorable, for some $r > 0$. Every 1-regular graph has $exc_{max}(G) = 0$, as it consists entirely of a 1-factor. G is a 2-regular graph with $exc_{max}(G) = 0$ if and only if G is an even cycle or disjoint pair of even cycles. An even cycle has exactly two 1-factors and a disjoint pair of even cycles has exactly four 1-factors, of which any three have redundancy. Theorem 2.2 in Section 2 and the fact that odd cycles have no 1-factors show that all other 2-regular graphs have positive maximum excess.

A few examples of cubic graphs G with $exc_{max}(G) = 0$ are shown in Figure 1. It is immediate to see that $exc_{max}(K_4) = 0$ and $exc_{max}(Z_1) = 0$, as they both have exactly three 1-factors. It is quick to check by exhaustion that any 1-factor cover of Z_3 or $K_{3,3}$ of size at least 4 is not minimal. Therefore $exc_{max}(Z_3) = exc_{max}(K_{3,3}) = 0$.

Having a small number of 1-factors or unique 1-factorization does not imply that $exc_{max}(G) = 0$. There are exactly 17 graphs with fewer than $\frac{n}{2} + 2$ 1-factors [4]. Three of these graphs have $exc_{max}(G) \geq 1$. Examples of 1-factor covers of two these graphs are shown in Figure 2. The Petersen

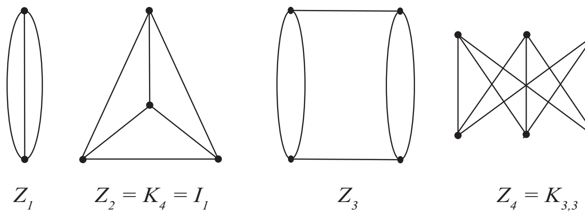

Figure 1: Small graphs with $exc_{max}(G) = 0$.

graph is the third graph and all of its minimal 1-factor covers have size 5. The other 14 graphs and their respective labels from [4] (I_j and H_j) appear in Figures 1, 3–5.

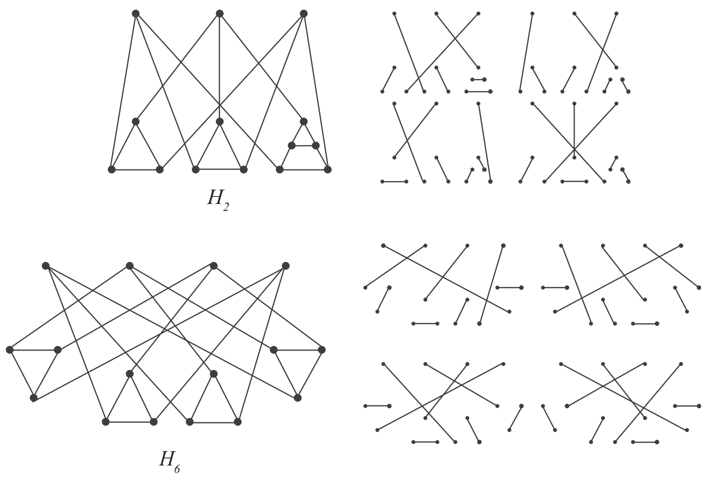

Figure 2: Examples of excessive factorizations that are not 1-factorizations.

In Section 2, we give some conditions on G that assure $exc_{max}(G) > 0$. We follow this by classifying all cubic graphs with $exc_{max}(G) = 0$ in Section 3. There are 24 such graphs, denoted by $\mathcal{Z} = \{Z_1, \ldots, Z_{24}\}$. Those not shown in Figure 1 are shown in Figures 3–5. Section 4 addresses the maximum excess of r-regular graphs with $r > 3$. In particular, we classify all bipartite graphs G with $exc_{max}(G) = 0$ and limit the number of r-regular graphs G with $exc_{max}(G) = 0$. Our results answer some questions of Wallis stated in [7]. We conclude with our own questions in Section 5.

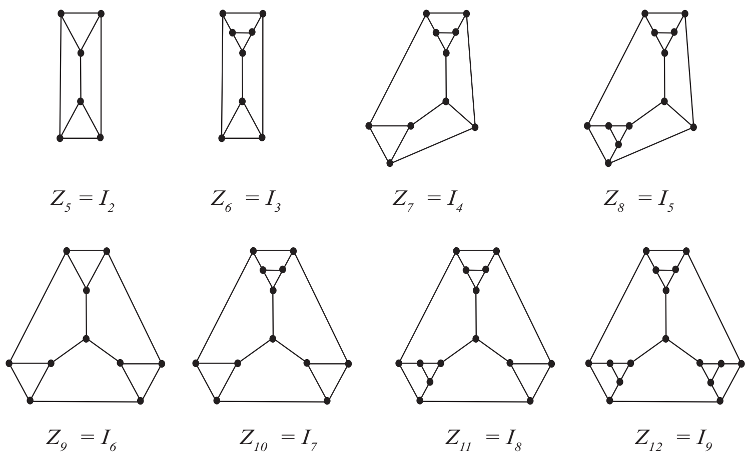

Figure 3: Graphs with $exc_{max}(G) = 0$ and $|EF(G)| = 3$.

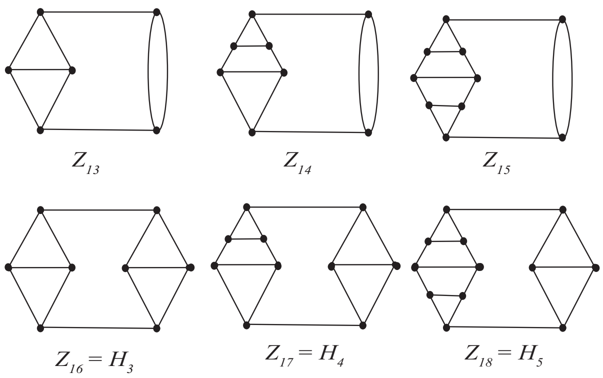

Figure 4: Graphs with $exc_{max}(G) = 0$ and $|EF(G)| = 5$.

2 Positive lower bounds for $exc_{max}(G)$

In this section, we examine structural properties of r-regular graphs G ($r \geq 3$) which imply $exc_{max}(G) > 0$. Since the focus of this paper is on r-regular graphs G with $exc_{max}(G) = 0$, all graphs should be considered connected unless otherwise noted. Theorem 2.2 shows that if G is disconnected then $exc_{max}(G) > 0$.

Lemma 2.1. *Let G be an r-regular graph and F be a 1-factor of G. If*

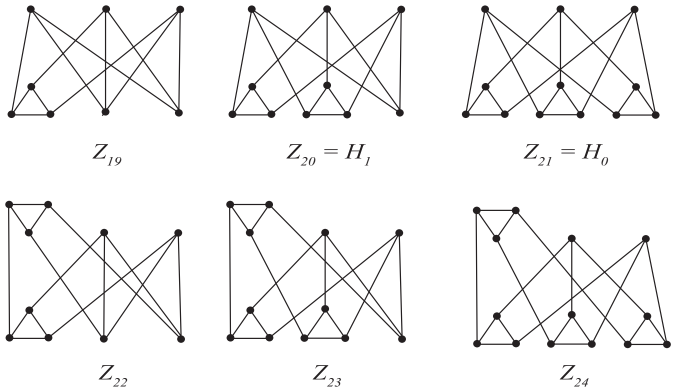

Figure 5: Graphs with $exc_{max}(G) = 0$ and $|EF(G)| = 6$.

$exc_{max}(G) = 0$, then $exc_{max}(G - F) = 0$ or ∞.

Proof. Assume $exc_{max}(G-F) \neq 0, \infty$. Then there exists an integer $k > r-1$ such that $G - F$ has a minimal 1-factor cover $\mathcal{F}$ such that $|\mathcal{F}| = k$. Then $\mathcal{F} \cup F$ is a minimal 1-factor cover of G and $|\mathcal{F} \cup F| > r$. This implies $exc_{max}(G) \neq 0$, which is a contradiction. $\square$

Theorem 2.2. *Let G be an r-regular graph with $k > 0$ components, $G_1, G_2, \ldots, G_k$. If $\chi'_e(G_i) < \infty$ for all $1 \leq i \leq k$, then $exc_{max}(G) \geq (k-1)(r-1) - 1$.*

Proof. Each G_i has a 1-factor cover of size r_i, where $r_i \geq \chi'_e(G_i)$. Label these 1-factors $F_{(i,1)}, \ldots, F_{(i,r_i)}$. For each (i, j), let

$$F'_{(i,j)} = F_{(i,j)} \cup \bigcup_{\substack{\ell=1 \\ \ell \neq i}}^{k} F_{(\ell,1)}$$

and define $\mathcal{F} = \{F'_{(i,j)} \mid 1 \leq i \leq k \text{ and } 2 \leq j \leq r_i\}$. Each $F'_{(i,j)}$ is a 1-factor of G and there are $\sum_{i=1}^{k}(r_i - 1)$ such 1-factors in $\mathcal{F}$, each of which has an edge that is not covered by any of the others. So $\mathcal{F}$ is a minimal 1-factor cover of G. Therefore, $exc_{max}(G) \geq \sum_{i=1}^{k} r_i - k - r$. Since $r_i \geq r$, $exc_{max}(G) \geq kr - k - r = (k-1)(r-1) - 1$. $\square$

Lemma 2.3. *Let G be an r-regular graph with distinct 1-factors F_1 and F_2. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be disjoint sets of 1-factors of G such that $|\mathcal{F}_1| = |\mathcal{F}_2| = r-1$. If $\{F_1\} \cup \mathcal{F}_1$, $\{F_2\} \cup \mathcal{F}_2$, and $\{F_1\} \cup \{F_2\} \cup \mathcal{F}_3$ are all 1-factorizations of G, then $\mathcal{F}_1 \cup \mathcal{F}_2$ is a minimal 1-factor cover of G. Moreover, $exc_{max}(G) \geq r-2$.*

Proof. Clearly, the set of edges covered by $\mathcal{F}_1$ is the same as the set of edges covered by $\{F_2\} \cup \mathcal{F}_3$ and the set of edges covered by $\mathcal{F}_2$ is the same as the set of edges covered by $\{F_1\} \cup \mathcal{F}_3$. Therefore, $\mathcal{F}_1 \cup \mathcal{F}_2$ is a 1-factor cover of G. $\mathcal{F}_1 \cup \mathcal{F}_2$ is also minimal, because removing a 1-factor of $\mathcal{F}_2$ from $\mathcal{F}_1 \cup \mathcal{F}_2$ would leave some edge contained in F_1 uncovered. If not, then $\{F_1\} \cup \{F_2\} \cup \mathcal{F}_3$ is not minimal. Similarly, removing a 1-factor of $\mathcal{F}_1$ from $\mathcal{F}_1 \cup \mathcal{F}_2$ would leave some edge contained in F_2 uncovered. Hence, $\mathcal{F}_1 \cup \mathcal{F}_2$ is a minimal 1-factor cover and $exc_{max}(G) \geq r-2$. □

3 Cubic graphs with $exc_{max}(G) = 0$

In this section we will prove that $\mathcal{Z}$ is the set of all cubic graphs with $exc_{max}(G) = 0$, where $\mathcal{Z}$ is the set of graphs introduced in Section 1. We begin by showing $exc_{max}(G) = 0$ for every $G \in \mathcal{Z}$. We will go on to show that if $exc_{max}(G) = 0$, then $G \in \mathcal{Z}$. Note that every graph in $\mathcal{Z}$ has at most $\frac{n}{2} + 3$ 1-factors. For several of the following results, we will need hypergraphs and edge cuts, so we give the necessary definitions here.

A *hypergraph* is an ordered pair $H = (V, E)$, where V is a set of vertices and E is a collection of hyperedges. A *hyperedge* is a subset of V. We denote the set of vertices of H by $V(H)$. The degree of a vertex $v \in V(H)$ is the number of hyperedges containing v.

A *transversal* of a hypergraph H is a subset T of $V(H)$ such that $h \cap T \neq \emptyset$ for every $h \in E(H)$. A transversal is *minimal* if it has no proper subset that is also a transversal. The *blocker of* H, denoted by $b(H)$, is the set of all minimal transversals of H. $b(H)$ is also a hypergraph, with the same vertex set as H. A *clutter* is a hypergraph with the property that no hyperedge properly contains another. It is well known that if H is a clutter, then $b(H)$ is also a clutter and $b(b(H)) = H$ [3].

One of our techniques is to translate the problem about 1-factor covers of graphs into a problem about transversals of hypergraphs. We start by creating a representative hypergraph for each graph; an example is given in Figure 6.

For each graph G, we define a hypergraph H_G, where $V(H_G)$ is the set of all 1-factors of G and for each edge $e \in E(G)$, there is a hyperedge $h_e \in H_G$ that is the set of 1-factors containing e. We then define H'_G as the hypergraph with $V(H'_G) = V(H_G)$ and $E(H'_G)$ being the set of hyperedges of H_G that do not properly contain any hyperedge of H_G. It is important to note that H_G may be a multiset; however, H'_G cannot. Therefore, H'_G is a clutter. Moreover, any transversal of H'_G is a 1-factor cover of G and any minimal transversal of H'_G is a minimal 1-factor cover of G.

H_G	H'_G	$b(H'_G)$
$\{F_8, F_9\}$, $\{F_6, F_7, F_9\}$, $\{F_1, F_2, F_6, F_9\}$	$\{F_2, F_3\}$, $\{F_7, F_3\}$	$\{F_2, F_7, F_8\}$
$\{F_7, F_9\}$, $\{F_4, F_5, F_8\}$, $\{F_4, F_5, F_6, F_7\}$	$\{F_2, F_5\}$, $\{F_7, F_5\}$	$\{F_3, F_5, F_9\}$
$\{F_5, F_8\}$, $\{F_1, F_2, F_3\}$, $\{F_1, F_3, F_4, F_8\}$	$\{F_2, F_9\}$, $\{F_7, F_9\}$	
$\{F_5, F_7\}$, $\{F_2, F_3\}$, $\{F_1, F_4, F_5, F_6, F_7\}$	$\{F_8, F_3\}$	
$\{F_2, F_5\}$, $\{F_3, F_8\}$, $\{F_1, F_2, F_4, F_6, F_9\}$	$\{F_8, F_5\}$	
$\{F_3, F_7\}$, $\{F_2, F_9\}$, $\{F_1, F_3, F_4, F_6, F_8\}$	$\{F_8, F_9\}$	
$\{F_3, F_4, F_7\}$, $\{F_2, F_5, F_6\}$, $\{F_1, F_8, F_9\}$		

Figure 6: Example of H_G, H'_G, and $b(H'_G)$ when $G = Z_{24}$.

An *m-edge cut* of G is a set of m edges that disconnect G when removed. An edge cut M is nontrivial if there is no vertex v such that every edge of M is incident with v.

Suppose G is a cubic 3-edge colorable graph and M is a 3-edge cut of G. Let C_1 and C_2 be the two components of $G - M$. For $i = 1, 2$ let $G_i(M)$ be the graph obtained from G by contracting C_i to a single vertex. As a matter of convenience, we will refer to $G_i(M)$ as G_i when M is not ambiguous. G_1 and G_2 are both cubic and by Lemma 3.1 both are 3-edge colorable. In many instances we will only be concerned with exactly one of the contractions. In such cases, without loss of generality, we will say $G_M = G_1$.

Lemma 3.1. *Let G be a 3-edge colorable cubic graph. If F is an extendable 1-factor and M is a 3-edge cut of G, then $|F \cap M| = 1$.*

Proof. Recall that a 1-factor of G contained in some 1-factorization of G is an extendable 1-factor. If F is an extendable 1-factor, then the removal of F must leave even cycles. This is not possible if $|F \cap M|$ is 0 or 2, since $G - F$ would have an odd sized edge cut. Let C_1 and C_2 be the two components of

$G - M$. If $|F \cap M| = 3$, then $|V(C_1)|$ and $|V(C_2)|$ are both even. However, this results in contradiction since C_1 and C_2 will have an odd number of vertices of degree 3. □

Lemma 3.2. *Let G be a 3-edge colorable cubic graph with 3-edge cut M. Then $exc_{max}(G_M) \leq exc_{max}(G)$.*

Proof. Let $M = \{m_1, m_2, m_3\}$ and $\{F_1, F_2, F_3\}$ be a 1-factorization of G such that $m_i \in F_i$. Let F_M be a 1-factor of G_M and without loss of generality, let $m_1 \in F_M$. Then $F_M \cup (F_1 - E(G_M))$ is a 1-factor of G. This defines an injection ϕ from the 1-factors of G_M to the 1-factors of G. Therefore, if $\mathcal{F_M}$ is a minimal 1-factor cover of G_M, then $\mathcal{F} = \{\phi(F) | F \in \mathcal{F_M}\}$ is a minimal 1-factor cover of G. If $|\mathcal{F_M}| = exc_{max}(G_M)$, then $exc_{max}(G_M) \leq exc_{max}(G)$. □

Lemma 3.3. *Let G be a 3-colorable cubic graph. If $G \in \mathcal{Z}$, then $exc_{max}(G) = 0$.*

Proof. For each $G \in \mathcal{Z}$ there exists a sequence of graphs in $\mathcal{Z}$, $G = G_0$, G_1, $G_2, \ldots,$ G_k, such that G_{i-1} is attained by contracting a K_3 of G_i for $1 \leq i \leq k$, and G_k is Z_{12}, Z_{18}, or Z_{24}. Therefore, because of Lemma 3.2, we only need to show $exc_{max}(Z_{12})$, $exc_{max}(Z_{18})$, and $exc_{max}(Z_{24})$ all equal 0.

It can easily be seen in Figure 6 that $exc_{max}(Z_{24}) = 0$. There exist three edges in Z_{12} such that each edge is contained in exactly one 1-factor (see Figure 7), and no two of these edges are in the same 1-factor [4]. Since Z_{12} is 3-edge colorable, $exc_{max}(Z_{12}) = 0$.

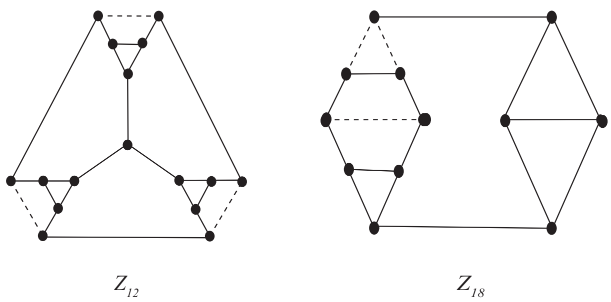

Figure 7: Edges of Z_{12} and Z_{18} that determine excessive factorizations.

Similarly, Figure 7 illustrates three edges of Z_{18} in which one edge is covered by exactly one 1-factor of Z_{18} and each of the other two edges are covered by exactly two 1-factors of Z_{18}. These five 1-factors of Z_{18} obviously form a 1-factor cover of Z_{18} and only contain two minimal 1-factor covers of Z_{18}, both of size three.

□

Theorem 3.4. *If G is a 3-edge colorable cubic graph on n vertices such that every 1-factor is extendable and $exc_{max}(G) = 0$, then G is K_4, $K_{3,3}$, Z_1, Z_3, Z_{13}, Z_{16}, Z_{19}, Z_{20}, or Z_{21}.*

Proof. In order to prove by contradiction we will assume G is not one of the previously listed graphs. It is an easy check that G is not a graph in $\mathcal{Z}$, which implies that there are at least $\frac{n}{2} + 2$ 1-factors of G. We will also assume $n \geq 6$, since every cubic graph on 2 or 4 vertices is in $\mathcal{Z}$.

By our previously stated construction, $b(H'_G)$ is the set of all 1-factorizations of G. If there exists a vertex $v \in V(b(H'_G))$ with $deg(v) \geq 3$, then the removal of the corresponding 1-factor of G, F_v, leaves a 2-regular subgraph that has at least 3 different 1-factorizations. This implies $G - F_v$ is a union of at least 3 disjoint even cycles. By Theorem 2.2, $exc_{max}(G) \geq 1$.

Therefore, each vertex of $b(H'_G)$ must have degree 1 or 2. If any hyperedge of $b(H'_G)$ contains two vertices of degree 2 (say a and b), $b(H'_G)$ must include hyperedges $\{a, b, c\}, \{a, d, e\}$, and $\{b, f, g\}$. It then follows from Lemma 2.3 that there can be at most 1 vertex of degree 2 in any hyperedge of $b(H'_G)$. Thus, each component of $b(H'_G)$ is either of type (1), namely a single hyperedge with three vertices of degree 1, or of type (2), namely two hyperedges each having two vertices of degree 1 and a common vertex of degree 2.

Let t be the number of components of type (2) and let $|\mathcal{F}| = |V(H'_G)|$ be the number of 1-factors of G. Then, because there are 5 vertices in each component of type (2), there are $\frac{|\mathcal{F}|-5t}{3}$ components of type (1). Therefore, $|b(b(H'_G))| = 5^t 3^{\frac{|\mathcal{F}|-5t}{3}}$, and this equals $|H'_G|$ as well because H'_G is a clutter. So $\frac{3n}{2} = |E(G)| = |H_G| \geq |H'_G| = 5^t 3^{\frac{|\mathcal{F}|-5t}{3}}$. Since $|\mathcal{F}| \geq 5t$, $5^t 3^{\frac{|\mathcal{F}|-5t}{3}} \geq 5^{\frac{|\mathcal{F}|}{5}}$.

By our assumption, $|\mathcal{F}| \geq \frac{n}{2} + 2$. Therefore, $\frac{3n}{2} \geq 5^{\frac{n}{10}+\frac{2}{5}}$, which implies $n \leq 15$. Since n must be even, $n = 6, 8, 10, 12$, or 14. The conditions $\{|\mathcal{F}| \geq \frac{n}{2} + 2, |\mathcal{F}| \geq 5t, |\mathcal{F}| - 5t \equiv 0 \bmod 3, \frac{3n}{2} \geq 5^t 3^{\frac{|\mathcal{F}|-5t}{3}}\}$ together have very few solutions.

The solution $n = 6$, $|\mathcal{F}| = 6$, and $t = 0$ only describes $K_{3,3}$. The solution $n = 6$, $|\mathcal{F}| = 5$, and $t = 1$ only describes Z_{13}. The solution $n = 8$, $|\mathcal{F}| = 6$, and $t = 0$ only describes Z_{19}. The only other solutions are $n = 10$ or 12,

$|\mathcal{F}| = 8$, and $t = 1$. We claim and will show by contradiction that there is no graph that satisfies either solution.

If there were such a graph G that satisfied either solution, then without loss of generality, $b(H'_G) = \{\{a,b,c\}, \{a,d,e\}, \{f,g,h\}\}$, so $H'_G = \{\{a,f\}, \{a,g\}, \{a,h\}, \{b,d,f\}, \{b,d,g\}, \{b,d,h\}, \{b,e,f\}, \{b,e,g\}, \{b,e,h\}, \{c,d,f\}, \{c,d,g\}, \{c,d,h\}, \{c,e,f\}, \{c,e,g\}, \{c,e,h\}\}$. Because H_G must be $\frac{n}{2}$-regular and b has degree 6, n can not be 10 and must be 12 and $H_G = \{\{a,f\}, \{a,f\}, \{a,g\}, \{a,g\}, \{a,h\}, \{a,h\}, \{b,d,f\}, \{b,d,g\}, \{b,d,h\}, \{b,e,f\}, \{b,e,g\}, \{b,e,h\}, \{c,d,f\}, \{c,d,g\}, \{c,d,h\}, \{c,e,f\}, \{c,e,g\}, \{c,e,h\}\}$.

Since a has degree 2 in $b(H'_G)$, the removal of F_a from G must leave exactly 2 disjoint even cycles, A_1 and A_2. Both A_1 and A_2 must be cycles on 6 vertices. From H_G we see that a contains 2 edges, f_1 and f_2, of F_f. In order for F_f to be a 1-factor of G, those 2 edges must either be incident with two adjacent vertices of A_1 or incident with two nonadjacent vertices of A_1 that have no common neighbors. Similar conditions hold for the vertices in A_2 that are incident with edges f_1 and f_2.

From $b(H'_G)$, we see that each edge of A_1 is in exactly one of the 1-factors F_b and F_c, and exactly one of the 1-factors F_d and F_e. If f_1 and f_2 are incident to adjacent vertices then the two edges of F_f in A_1 must both be in F_b or F_c and both be in F_d or F_e. This contradicts the structure of H_G.

Therefore, each pair of edges in $F_a \cap F_f$, $F_a \cap F_g$, and $F_a \cap F_h$ must be incident with two nonadjacent vertices in A_1 and incident with two nonadjacent vertices in A_2. This implies there are two edges u_1 and u_2 that are incident with adjacent vertices in A_1 and incident with adjacent vertices in A_2. Thus, u_1 and u_2, along with 2 edges of A_1 and 2 edges of A_2, form a new 1-factor of G, not listed in H_G. Therefore, no such G with $n = 12$, $t = 1$, and $|F| = 8$ exists. □

Lemma 3.5. *Suppose G is a 3-edge-colorable cubic graph with no nontrivial 3-edge cut. If G has a non-extendable 1-factor F, then $exc_{max}(G) \geq 1$.*

Proof. Let p be a positive integer. By [6, (2.3)], there exists a multiset $\mathcal{F}$ of 1-factors of G such that each edge of G is contained in exactly p (not necessarily distinct) 1-factors of $\mathcal{F}$. Let $e \in E(G)$ be an edge contained in F and $\mathcal{F}_e = \mathcal{F} - \{F_i \mid e \in F_i \text{ and } F \neq F_i\}$. Since at most $(p-1)$ 1-factors of $\mathcal{F}$ are removed, $\mathcal{F}_e$ is a 1-factor cover of G. So there exists a minimal 1-factor cover of G that is a subset of $\mathcal{F}_e$ and it must contain F. Hence, there exists a minimal 1-factor cover of G that is not a 1-factorization. Thus, $exc_{max}(G) \geq 1$. □

Corollary 3.6. *If G is a cubic graph, has a non-extendable 1-factor and $exc_{max}(G) = 0$, then G has a nontrivial 3-edge cut.*

Lemma 3.7. *K_4, $K_{3,3}$, Z_1, and Z_3 are the only cubic graphs G with no nontrivial 3-edge cut and $exc_{max}(G) = 0$.*

Proof. Assume G is a cubic graph with no nontrivial 3-edge cut and $exc_{max}(G) = 0$. If G has a non-extendable 1-factor, then Lemma 3.5 implies that $exc_{max}(G) \geq 1$. So every 1-factor of G is extendable, and by Theorem 3.4 G is K_4, $K_{3,3}$, Z_3, or Z_1.

□

Theorem 3.8. *If G is a cubic graph and $exc_{max}(G) = 0$, then the following are true:*

i. $3 \leq |EF(G)| \leq 6$.

ii. If M is a nontrivial 3-edge cut of G, then there exists a simple G_M such that $|EF(G_M)| = 3$.

iii. If $G \neq K_{3,3}$, Z_1, or Z_3, then K_3 is a subgraph of G.

Proof. We will prove a statement slightly stronger than (i) by induction on the number of nontrivial 3-edge cuts in G, namely that if G is a cubic graph and $exc_{max}(G) = 0$, then $|EF(G)|$ is 3, 5, or 6. The graphs of Lemma 3.7 will be the base case. Clearly, $|EF(K_4)| = 3$, $|EF(K_{3,3})| = 6$, $|EF(Z_1)| = 3$, and $|EF(Z_3)| = 5$. If G is not one of these graphs, then G has a nontrivial 3-edge cut, $M = \{m_1, m_2, m_3\}$. Let $x_i = |EF_{G_1}(m_i)|$ and $y_i = |EF_{G_2}(m_i)|$, for $1 \leq i \leq 3$. So $|EF(G_1)| = \sum x_i$ and $|EF(G_2)| = \sum y_i$. Every nontrivial 3-edge cut of G_1 or G_2 is also a nontrivial 3-edge cut of G. Therefore, G_1 and G_2 have fewer nontrivial 3-edge cuts than G, so $|EF(G_1)|, |EF(G_2)| \in \{3, 5, 6\}$. Every extendable 1-factor of G must be the union of an extendable 1-factor of G_1, say F_1, and an extendable 1-factor of G_2, say F_2, such that $F_1 \cap F_2$ is m_1, m_2, or m_3. Therefore, $|EF(G)| = \sum x_i \cdot y_i$.

Let X and Y be the multisets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$, respectively. If $x_1 = 1$, then the removal of the extendable 1-factor of G_1 containing m_1 must leave a 2-regular graph G', and every edge of G', including m_2 and m_3, is on the same number of extendable 1-factors of G'. In fact, these extendable 1-factors of G' correspond to the only extendable 1-factors of G containing m_2 and m_3. Thus these edges, considered as part of G, are on the same number of extendable 1-factors. Therefore $X, Y \in \{\{1, 1, 1\}, \{1, 2, 2\}, \{2, 2, 2\}\}$.

Now consider the case that for some i, $x_i = y_i = 2$. Let A_1, A_2 be the extendable 1-factors in G_1 containing m_i and let B_1, B_2 be the corresponding extendable 1-factors in G_2 containing m_i. $A_1 \cup B_2$ is an extendable 1-factor of G and thus forms a 1-factorization of G with 1-factors C_1, C_2. The four 1-factors of G $\{A_1 \cup B_1, A_2 \cup B_2, C_1, C_2\}$ form a minimal 1-factor cover of G that is not a 1-factorization, so that $exc_{max}(G) \geq 1$; this is a contradiction. Therefore at least one of X or Y is $\{1, 1, 1\}$. Therefore, $|EF(G)| = \sum y_i$. This proves part (i) and shows that $|EF(G_1)| = 3$. To prove (ii) we only need to show G_1 is simple.

We claim that if $G_1 \neq Z_1$ is not simple, then $|EF(G_1)| = 5$. Assume $u, v \in V(G_1)$ and $e_1, e_2 \in E(G_1)$, such that e_1 and e_2 are both incident with u and v. Let $\{F_1, F_2, F_3\}$ be a 1-factor cover of G_1 such that $e_1 \in F_1$ and $e_2 \in F_2$. Then $\{F_1 - \{e_1\} \cup \{e_2\}, F_2 - \{e_2\} \cup \{e_1\}, F_3\}$ is also a minimal 1-factor cover of G_1. Therefore, $|EF(G_1)| \geq 5$. However, $|EF(G_1)| \neq 6$, as otherwise Lemma 2.3 would yield a contradiction.

Finally, we prove (iii) by contradiction. Let G be the smallest cubic graph $G \neq K_{3,3}$, Z_1, or Z_3 with $exc_{max}(G) = 0$ that does not contain K_3 as a subgraph. Then G has a nontrivial 3-edge cut M_1 such that G_{M_1} is simple and $|EF(G_{M_1})| = 3$. So G_{M_1} is not $K_{3,3}$, Z_1, or Z_3 and G_{M_1} has fewer vertices than G, so G_{M_1} must contain K_3.

Let $L_1, R_1 \subset V(G)$ be the partition of vertices attained from the components of $G - M_1$. Since G_{M_1} contains K_3 and G does not, two edges of M_1 must be incident to adjacent vertices u_1 and v_1. Without loss of generality, let $u_1, v_1 \in L_1$. Let $L_2 = L_1 - \{u_1, v_1\}$, $R_2 = R_1 \cup \{u_1, v_1\}$, and M_2 be the set of edges between L_2 and R_2 (as seen in Figure 8). Now contracting R_2 in G is equivalent to contracting the triangle created in G_{M_1}. Therefore, M_2 is a nontrivial 3-edge cut such that K_3 is a subgraph of G_{M_2} and $|EF(G_{M_2})| = 3$.

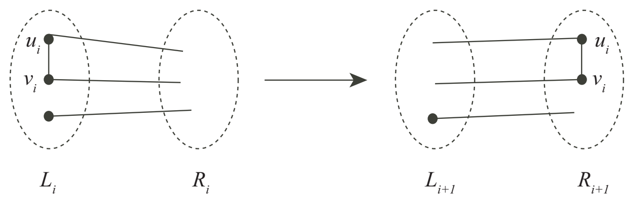

Figure 8: Augmenting sets L_i and R_i.

Since G is finite, this process must terminate. If it terminates at the kth iteration, then $|L_k| = 3$. If $|L_k| = 3$ and G is cubic, then K_3 is a subgraph of

L_k, which is a contradiction to K_3 not being a subgraph of G. $\square$

Theorem 3.9. *Let G be a 3-colorable cubic graph. Then $G \in \mathcal{Z}$ if and only if $exc_{max}(G) = 0$.*

Proof. We will prove the converse of Lemma 3.3 by contradiction. Assume G^* is the smallest simple cubic graph with $exc_{max}(G^*) = 0$ such that G^* is not a graph in $\mathcal{Z}$. Then G^* must contain K_3 as a subgraph and contracting a K_3 in G^* must yield a graph contained in $\mathcal{Z}$. However, for each $G \in \mathcal{Z}$, it can be easily checked exhaustively that replacing a vertex with a K_3 renders a graph $G_\triangle$ where $G_\triangle \in \mathcal{Z}$ or $exc_{max}(G_\triangle) \geq 1$. $\square$

4 Regular Graphs with $exc_{max}(G) = 0$

Whereas 3-regular graphs G with $exc_{max}(G) = 0$ were classified in Section 3, here we consider r-regular graphs G for $r > 3$ with $exc_{max}(G) = 0$. The following lemma shows that there are finitely many such simple graphs.

Theorem 4.1. *If G is an r-regular graph ($r > 3$) on $n > 16$ vertices, then $exc_{max}(G) \geq 1$.*

Proof. In order to prove by contradiction, assume $exc_{max}(G) = 0$. Then G has a 1-factorization $\{F_1, F_2, \ldots, F_r\}$. Let $G' = (V(G), F_1 \cup F_2 \cup F_3)$. Then G' is a cubic graph that is not contained in $\mathcal{Z}$. However, by Lemma 2.1, $exc_{max}(G') = 0$. This is a contradiction. $\square$

Theorem 4.2. *The only simple regular bipartite graph with $exc_{max}(G) = 0$ is $K_{3,3}$.*

Proof. Let G be a simple r-regular bipartite graph. It has been shown that if $r = 3$, then G is $K_{3,3}$. Let $r = 4$ and F be a 1-factor of G. $G - F$ is a 3-regular simple bipartite graph. Lemma 2.1 implies $G - F$ must be $K_{3,3}$; so G is K_6 minus a 1-factor, and $exc_{max}(G) \geq 1$. Now let $r > 4$ and F be a 1-factor of G. Then $exc_{max}(G - F) \geq 1$, which implies $exc_{max}(G) \geq 1$. $\square$

5 Conclusion

The *excess range* of a graph G is the interval $[exc(G), exc_{max}(G)]$. We say that the excess range is empty if and only if $\chi'_e(G) = \infty$. We have only investigated those regular graphs with excess range $[0,0]$. The results of this paper on regular graphs G with $exc_{max}(G) = 0$ lead to more general questions about the excess range of regular graphs. The Petersen Graph has excess range $[2,2]$. Is it the only cubic graph with excess range $[2,2]$? Do there exist regular graphs with excess range $[k,k]$ for any other positive k?

In contrast to graphs with singleton excess range, there also exist graphs with arbitrarily large excess range (consider, for example, $C_{2n} \times K_2$ and apply Theorem 2.2). Does there exist, for each positive k, a graph with excess range $[0,k]$?

Finally, we have found some graphs (by replacing a vertex in each of Z_{23} and Z_{24} with a K_3) that have gaps in their excess ranges. Are most regular graphs free of gaps in their excess ranges, or is this relatively uncommon? Can infinite families of such graphs be constructed? This last set of questions seems the most challenging to investigate.

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